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Asymptotic Limits in the Hitchin moduli space

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Asymptotic Limits in the Hitchin moduli space

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Dedicated to my parents.

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Asymptotic Limits in the Hitchin moduli space

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Given a Higgs bundle $(\bar{\partial}_E, \varphi)$, Hitchin's equations are equations for a Hermitian metric on the underlying vector bundle. Hitchin's equations are a coupled system of non-linear PDEs, and as such are difficult to solve. Despite this, there are some situations in which it is possible to say something more concrete about the Hermitian metric solving Hitchin's equations. This is the unifying theme of this three-part dissertation.

In the first chapter, we look at the ends of the Hitchin moduli space on a compact Riemann surface. We construct good approximate solutions of Hitchin's equations near the ends, taking advantage of the asymptotic abelianization of Hitchin's equations.

In the second chapter, we consider solutions of Hitchin's equations on $\mathbb{CP}^1$ which are fixed by a circle action. The circle action manifests in a radial symmetry which reduces Hitchin's equations from a coupled system of PDEs to a coupled system of ODEs. In the main result of the second part, we relate fixed points of the circle

action to $\mathcal{W}$-algebra representations. We prove that for each representation in the $(K, K+N)$-minimal model of $\mathcal{W}_K$, the effective central charge is equal to a number which can be computed from a solution of Hitchin's equations fixed by a certain circle action.

The Hitchin moduli space has two interesting subspaces: the Hitchin section and the space of opers. In the third part, we relate two different families of flat connections corresponding to these two subspaces. To relate these families, we study how a certain harmonic metric blows up. This harmonic metric is related to the uniformizing metric on the underlying complex curve.

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Generic Ends of the Moduli Space of $\mathrm{SL}(n, \mathbb{C})$-Higgs Bundles

1.1 Introduction

Hitchin's equations [Hit87] are a system of gauge theoretic equations on a Riemann surface. Hitchin moduli spaces seem to be very important objects in geometry. They are studied under a number of different names including moduli spaces of Higgs bundles and $SL(n, \mathbb{C})$-character varieties of surface groups. They are featured in Kapustin-Witten's interpretation of the geometric Langlands correspondence [KW06, HT02] and are of interest in mirror symmetry [Wit08], quantization [BD91], and Teichmüller theory [Hit92, Gol08]. Hitchin's equations arise in physics in a number of ways [Wit09, Moo12], and many of these links with other areas of mathematics have been inspired by physical dualities.

The Hitchin moduli space is a non-compact manifold. It is a complex integrable system, and as such algebro-geometric tools have been powerful tools for studying the Hitchin moduli space. This complex integrable system structure manifests in a fibration $\mathcal{M}$, shown in Figure 1.1.

This paper concerns the non-compact ends of the Hitchin moduli space $\mathcal{M}$, indicated in Figure 1.1 by the gray region. Many conjectures in mathematics and

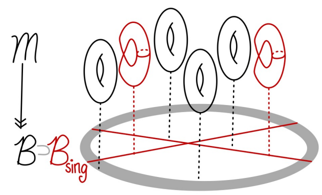

Figure 1.1: *The complex integrable system structure manifests in a Lagrangian fibration of* $\mathcal{M}$. *Generic fibers are compact complex tori. The ends of the moduli space are indicated in gray.*

physics about these ends are open because they require a finer knowledge of the ends of the moduli space than traditional algebro-geometric techniques provide. However, a number of results [MSWW14, MSWW15, Moc15] in the last two years demonstrate the power of constructive analytic techniques for describing the "ends" of the Hitchin moduli space. Finer knowledge of the non-compact ends starts with finer descriptions of solutions of Hitchin's equations near the "ends." In [MSWW14], Mazzeo-Swoboda-Weiss-Witt give explicit constructions of approximate solutions of Hitchin's equations away from the degenerate torus fibers when $G_{\mathbb{C}} = SL(2,\mathbb{C})$. This paper extends their construction to higher-rank groups $G_{\mathbb{C}} = SL(n,\mathbb{C})$— still staying away from the degenerate torus fibers. Many of the proofs in this paper are modeled on the proofs of Mazzeo-Swoboda-Weiss-Witt[MSWW14]. Consequently, we mostly use their conventions for comparative convenience.

For this paper, let $C = C(I, g_C, \omega)$ be a compact Kähler curve of genus ≥ 2. Let g_C be the conformal metric for the complex curve $C = C(I)$. Let $K = K_C$ be the canonical line bundle. Let $E \to C$ be a complex vector bundle of rank n and degree d. Let Det E be the determinant line bundle. We write $\mathrm{Aut}(E)$ and $\mathrm{End}(E)$

to indicate the automorphisms and endomorphisms of E which are the identity on Det E.

At times, we equip the complex vector bundle E with a hermitian or holomorphic structure. These structures should also induce a fixed structure on Det E. Consequently, at the start, fix a holomorphic structure, $\bar{\partial}_{\mathrm{Det}\ E}$, and a hermitian structure, $h_{\mathrm{Det}\ E}$, on on Det E. We additionally impose the following compatibility condition: $h_{\mathrm{Det}\ E}$ is Hermitian-Einstein for the holomorphic line bundle $(\mathrm{Det}\ E, \bar{\partial}_{\mathrm{Det}\ E})$. The Hermitian-Einstein condition states that associated Chern connection $D = D(\bar{\partial}_{\mathrm{Det}\ E}, h_{\mathrm{Det}\ E})$ on Det E is projectively flat. Its curvature, F_D, is related to the degree of E by

$$F_D = -\sqrt{-1}\mathrm{deg}E\frac{2\pi\omega}{\mathrm{vol}_g(C)}. \tag{1.1}$$

In this paper we are interested in describing solutions of Hitchin's equations near the ends of the moduli space. The non-abelian Hodge correspondence gives a diffeomorphism between two different moduli spaces: the moduli space of stable Higgs bundles and the moduli space of solutions of Hitchin's equations. Higgs bundles are holomorphic objects. An $SL(n,\mathbb{C})$-Higgs bundle is a pair $(\bar{\partial}_E, \varphi)$ consisting of of a holomorphic structure $\bar{\partial}_E$ on $E \to C$ (which induces the fixed holomorphic structure $\bar{\partial}_{\mathrm{Det}\ E}$ on Det E), and a $\bar{\partial}_E$-holomorphic endomorphism $\varphi \in \Omega^{1,0}(C, \mathrm{End}(E))$, called the "Higgs field. We'll usually take the perspective that the data of a solution of Hitchin's equations is a triple $(\bar{\partial}_E, \varphi, h)$ where $(\bar{\partial}_E, \varphi)$ is a $SL(n,\mathbb{C})$-Higgs bundle and h is a Hermitian metric on E (which induces the fixed hermitian structure $h_{\mathrm{Det}E}$ on Det E). In this perspective, given a Higgs bundle $(\bar{\partial}_E, \varphi)$, Hitchin's equations are equations for a special Hermitian metric h— the so called "harmonic metric"—on E.

Such a harmonic metric h exists if, and only if, the Higgs bundle $(\bar{\partial}_E, \varphi)$ is polystable. We want to construct this metric.

Definition 1.1.1. A triple $(\bar{\partial}_E, \varphi, h)$ of a Higgs bundle $(\bar{\partial}_E, \varphi)$ together with a Hermitian metric H on $E \to C$ is a **solution of the t-rescaled Hitchin's equations** if

$$F_{D(\bar{\partial}_E, h)} + t^2[\varphi, \varphi^{\dagger_h}] = -\sqrt{-1}\frac{\deg (E)}{\operatorname{rank}(E)}\operatorname{Id}_E\frac{2\pi\omega}{\operatorname{vol}_g(C)}. \tag{1.2}$$

where $D(\bar{\partial}_E, h)$ is the Chern connection, F_D is its curvature, $\varphi^{\dagger_h}$ is the adjoint of φ with respect to h (in local coordinates $\varphi^{\dagger_h} = h^{-1}\varphi^{\dagger}h$), ω is the Kähler form on C, and g is the Riemannian metric on C.

In this paper, all Higgs bundles will be "simple" (Definition 1.2.1). Higgs bundles are generically simple. (The space of non-simple Higgs bundles has complex codimension one inside inside the Higgs bundle moduli space.) Some advantage of considering only simple Higgs bundles are discussed in §1.2.

Goal: *For a fixed simple stable Higgs bundle $(\bar{\partial}_E, \varphi)$, understand the behavior of the family of harmonic metrics h_t solving the t-rescaled Hitchin's equations for $t \gg 0$.*

A standard technique to understand the behavior of the family of harmonic metrics h_t on E is

- finding appropriate parameters for the moduli space,
- constructing a family of approximate solutions, h_t^{app}, depending on these parameters, and then

- perturbing from approximate solutions h_t^{app} to actual solutions h_t using an implicit function theorem.

In this paper, we construct approximate hermitian metrics. We prove that these approximate hermitian metrics h_t^{app} are close to solving Hitchin's equations.

Proposition (cf. Proposition 1.4.1) *There exists positive constants c, δ such that for $t \gg 1$,*

$$\left\| \mathbf{F}_t(\bar{\partial}_E, \varphi, h_t^{\mathrm{app}}) \right\|_{L^2(C)} \leq ce^{-\delta t}. \tag{1.3}$$

In the main theorem, Theorem 1.5.1, we prove that we can perturb from an approximate solution h_t^{app} to an actual solution h_t.

Theorem (cf. Theorem 1.5.1) *There is a value $m > 0$, such that for t sufficiently large, there is a unique Hermitian γ_t satisfying $\|\gamma_t\|_{H^2(i\mathfrak{su}(E))} \leq t^{-m}$ such $\mathbf{F}_t^{\mathrm{app}}(\gamma) = 0$, i.e. $(A_t^{\exp(\gamma_t)}, \Phi_t^{\exp(\gamma_t)})$ solves Hitchin's equations.*

This paper is outlined as follows. In §1.2, we describe simple Higgs bundles. In §1.3 and §1.4, we move towards constructing approximate solutions. In §1.3, we construct candidate limiting configurations h_∞. In §1.4, we build approximate solutions h_t^{app} by desingularizing the candidate limiting configuration. Note that in the construction of the candidate limiting h_∞ we make a choice of parabolic structure. If our choice of candidate limiting configuration were wrong, these approximate solutions would not be close to actual solutions. In §1.5, we prove the main theorem,

Theorem 1.5.1. The existence of a harmonic metric h_t close to h_t^{app} is proved using a contraction mapping argument. The contraction mapping argument depends on an estimate on a linear operator. The estimate is used in §1.5, but for flow, the proof is delayed until §1.6.

1.2 Simple Higgs bundles

The definition of a simple $SL(n,\mathbb{C})$-Higgs bundle generalizes the usual definition of a simple $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle. A $SL(2,\mathbb{C})$-Higgs bundle $(\bar{\partial}_E, \varphi)$, is simple if $\mathrm{Det}(\varphi)$ has simple zeros. For $SL(n,\mathbb{C})$, we consider the discriminant section associated to φ rather than $\mathrm{Det}(\varphi)$.

Definition 1.2.1. Let $(\bar{\partial}_E, \varphi)$ be a Higgs bundle. Let $\{\lambda_1(p), \cdots, \lambda_n(p)\}$ be the eigenvalues of $\varphi(p)$. A Higgs field is **simple** if the discriminant section Δ_φ

$$\begin{aligned} \Delta_\varphi : C &\rightarrow K_C^{n^2-n} \\ p &\mapsto \prod_{1\leq i<j\leq n} (\lambda_i(p) - \lambda_j(p))^2 \end{aligned} \tag{1.4}$$

has only simple zeros.

Remark 1.2.1. For $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{Det}(\varphi) = \lambda_1\lambda_2 = -\frac{1}{4}\Delta_\varphi$. Consequently, for $SL(2,\mathbb{C})$, the two definitions of "simple Higgs bundle" agree.

Simple Higgs bundles have a number of simplifying advantages. Many of them are related to the Hitchin map. The Hitchin map *Hit* maps the moduli space of Higgs bundles onto the Hitchin base $\mathcal{B}$. A Higgs bundle with Higgs field φ is mapped to its eigenvalues, encoded in the characteristic polynomial char_φ of φ:

$$\begin{aligned} Hit: \quad \mathcal{M} &\to \mathcal{B} \\ (\bar{\partial}_E, \varphi) &\mapsto \mathrm{char}_\varphi. \end{aligned} \tag{1.5}$$

For $SL(n, \mathbb{C})$, the characteristic polynomial of φ is

$$\mathrm{char}_\varphi(x) = x^n + b_2 x^{n-2} + \cdots b_{n-1}x + b_n, \tag{1.6}$$

for coefficients $b_i \in H^0(C, K_C^i)$. Note that because we consider $G_{\mathbb{C}} = SL(n, \mathbb{C})$, there is no coefficient $b_1 \in H^0(C, K_C^1)$. Identifying the polynomial, $\mathrm{char}_\varphi(x)$, with its coefficients gives an identification

$$\mathcal{B} \cong \oplus_{i=2}^n H^0(C, K_C^i). \tag{1.7}$$

Remark 1.2.2. Because a point $b \in \mathcal{B}$ encodes the eigenvalues of any φ in the fiber $Hit^{-1}(b)$, it makes sense to say that $b \in \mathcal{B}$ is "simple" if the associated discriminant Δ_b has only simple zeros.

The spectral cover $\Sigma \xrightarrow{\pi} C$ is the ramified n:1-cover that is cut-out of the holomorphic cotangent bundle $K_C \to C$ by the equation

$$\Sigma = \{\lambda \in K_C : \mathrm{char}_\varphi(\lambda) = 0\} \tag{1.8}$$

The preimage of a point $p \in C$ is $\Sigma_p = \pi^{-1}(p) = \{\lambda_1, \cdots, \lambda_n\}$, the (unordered) set of n-eigenvalues of $\varphi(p)$. The spectral cover is ramified at the zeros of Δ_φ.

Notation. Let $Z = \Delta_\varphi^{-1}(0) \subset C$, and call Z "the ramification locus."

Proposition 1.2.1. *If $(\bar{\partial}_E, \varphi)$ is a simple stable SL_n-Higgs bundle then*

- *the spectral cover Σ is smooth,*
- *there are $2(n^2-n)(g-1)$ ramification points,*
- *at each ramification point p, exactly two eigenvalues are equal. Locally, without loss of generality, let these be $\lambda_1(p)$ and $\lambda_2(p)$. There is a local holomorphic coordinate z centered at $p \in C$ such that $(\lambda_1 - \lambda_2)^2 = 4z dz^2$.*

Proof. Smoothness: The smoothness of Σ is most straightforward from Donagi's cameral cover perspective rather than the spectral cover perspective. If all zeros of the discriminant Δ_φ are simple, then the cameral cover $\widehat{\Sigma}$ is smooth with simple Galois ramification. The smoothness of cameral cover implies the smoothness of the spectral cover.

Local holomorphic coordinate: Moreover, the simple ramification of the cameral cover implies the simple ramification of the spectral cover. Consequently, at each ramification point, exactly two eigenvalues are equal. Without loss of generality, let these by λ_1 and λ_2. Moreover, because the ramification is simple, $(\lambda_1 - \lambda_2)^2$ has a simple zero. It is a standard argument (for example, see [Mas86] p. 216) that if a holomorphic quadratic differential $(\lambda_1 - \lambda_2)^2$ has a simple zero at p, then there is a local holomorphic coordinate z such that

$$(\lambda_1 - \lambda_2)^2 = 4z dz^2. \tag{1.9}$$

Number of ramification points: The discriminant Δ_φ has $2(n^2-n)(g-1)$ zeros, counted with multiplicity, because it is a section of the line bundle $K_C^{n^2-n}$ of degree $2(n^2-n)(g-1)$. Because all zeros of Δ_φ are simple, the counts with and without multiplicity are the same, hence $\#\{\Delta_\varphi^{-1}(0)\} = 2(n^2-n)(g-1)$. $\square$

Given a simple stable Higgs bundle $(\bar{\partial}_E, \varphi)$, the family of approximate hermitian metrics h_t^{app} constructed in §1.3-1.4 will depend on spectral data $(\mathcal{L}, \Sigma)$ associated to $(\bar{\partial}_E, \varphi)$. The spectral cover $\Sigma \to C$ is one piece of this data. It encodes the eigenvalues of φ in a holomorphic geometric object. The other piece is a holomorphic line bundle $\mathcal{L} \to \Sigma$, which encodes the eigenline bundles of φ. In the Hitchin fibration, each simple $b \in \mathcal{B}$ can be interpreted as a smooth spectral cover; the corresponding fiber $Hit^{-1}(b)$ of the Hitchin fibration is a compact complex torus, which can be interpreted as a Prym variety $\mathrm{Prym}(\Sigma, C)$[BNR89].

Remark 1.2.3. Note that if we were considering $G_{\mathbb{C}} = GL(n, \mathbb{C})$-Higgs bundles rather than $SL(n, \mathbb{C})$-Higgs bundles, the fiber of the Hitchin map $Hit^{-1}(b)$ would be $\mathrm{Pic}_{d'}(\Sigma)$, rather than $\mathrm{Prym}(\Sigma, C) \subset \mathrm{Pic}_{d'}(\Sigma)$, where $d' = d - n(n-1)(g-1)$.

We will use the spectral data $(\mathcal{L}, \Sigma)$ to prove the following proposition about the existence of a nice trivialization of E near a ramification point p.

Proposition 1.2.2. *Let $(\bar{\partial}_E, \varphi)$ be a simple stable Higgs bundle. Let $p \in Z \subset C$. Let z be the local holomorphic coordinate centered at p, given by Proposition 1.2.1. There is a local holomorphic trivialization of E over a disk $\mathbb{D}$ centered at p such that the Higgs bundle $(\bar{\partial}_E, \varphi)$ is*

$$\bar{\partial}_E = \bar{\partial} \tag{1.10}$$

$$\varphi = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda_3 & & \\ & & & \cdots & \\ & & & & \lambda_n \end{pmatrix} dz + \begin{pmatrix} 0 & 1 & & & \\ z & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} dz$$

for $\lambda = \frac{\lambda_1+\lambda_2}{2}, \lambda_i \in H^0(\mathbb{D}, K)$.

Proof. Let $\{\lambda_1(p), \cdots \lambda_n(p)\}$ be an enumeration of the eigenvalues of $\varphi(p)$ such that $\lambda_1(p) = \lambda_2(p)$ and z be a local holomorphic coordinate (from Proposition 1.2.1) on a neighborhood $U \subset C$ such that $(\lambda_1 - \lambda_2)^2 = 4z dz^2$. Take a disk $\mathbb{D} \subset U$ centered at p without additional ramification points. Over $\mathbb{D}$, the eigenvalues $\lambda_3, \cdots, \lambda_n$ are holomorphic functions. For each λ_i ($i \geq 3$), choose a holomorphic non-vanishing section e_i of the corresponding eigenlinebundle.

The eigenvalues λ_1, λ_2 are not functions on $\mathbb{D}$ since they are multi-valued. However, on the ramified double cover $\widetilde{\mathbb{D}} \xrightarrow{\pi} \mathbb{D}$ this multi-valued-ness resolves, and $\pi^*\lambda_1$ and $\pi^*\lambda_2$ are honest holomorphic functions. Define $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$. Let w be the holomorphic coordinate on $\widetilde{\mathbb{D}}$ such that $w^2 = z$. The coordinate w is defined up to sign, so without loss of generality, we may assume that we choose the sign and such that $\pi^*\lambda_1 = \pi^*\lambda + wd(w^2)$ and $\pi^*\lambda_2 = \pi^*\lambda - wd(w^2)$. Let σ be the involution of $\widetilde{\mathbb{D}}$ which exchanges the two sheets. In the holomorphic coordinate w, this corresponds to the map $\sigma : w \to -w$. Note that $\sigma^*\pi^*\lambda_1 = \pi^*\lambda_2$.

On this double cover, choose a single non-vanishing holomorphic section s_1 of the associated eigenline to $\pi^*\lambda_1$. Define $s_2 = \sigma^*(s_1)$. Note that s_2 is a non-vanishing holomorphic section of the associated eigenline to $\pi^*\lambda_2$.

Then in the basis $\{s_1, s_2, \pi^* e_3, \cdots, \pi^* e_n\}$, on $\widetilde{\mathbb{D}}$ the Higgs field $\pi^*\varphi$ is

$$\pi^*\varphi = \begin{pmatrix} \pi^*\lambda + wd(w^2) & & & & \\ & \pi^*\lambda - wd(w^2) & & & \\ & & \pi^*\lambda_3 & & \\ & & & \cdots & \\ & & & & \pi^*\lambda_n \end{pmatrix} \tag{1.11}$$

Because λ_1, λ_2 are not well-defined functions on $\mathbb{D}$, the basis s_1, s_2 does not make sense on $\mathbb{D}$. However, the basis

$$\begin{aligned} s_1' &= ((\pi^*\lambda_1 - \pi^*\lambda)s_1 + (\pi^*\lambda_2 - \pi^*\lambda)s_2) \\ s_2' &= (s_1 + s_2) \end{aligned} \tag{1.12}$$

satisfies $\sigma^* s_i' = s_i'$, consequently, it makes sense on $\mathbb{D} \subset C$. Let e_i' be the basis on $\mathbb{D} \subset C$ such that $\pi^* e_i' = s_i'$. One can check that

$$\begin{aligned} \varphi(e_1') &= \lambda e_1' + zdz^2 e_2' \\ \varphi(e_2') &= e_1' + \lambda e_2' \end{aligned} \tag{1.13}$$

Twist e_i' by dz to get a basis e_i in which

$$\begin{aligned} \varphi(e_1) &= (\lambda e_1 + z e_2)dz \\ \varphi(e_2) &= (e_1 + \lambda e_2)dz. \end{aligned} \tag{1.14}$$

Consequently, the given holomorphic trivialization of E exists. $\square$

Notation. By shrinking $\mathbb{D}$, we may assume that the disks around different points of Z do not intersect. By shrinking $\mathbb{D}$ further, we may assume that the difference between the eigenvalues of φ is bounded below by some positive constant $\varepsilon_\lambda > 0$ on

$C - \cup_{p\in Z}\mathbb{D}_p$. By possibly taking a smaller ε_λ, we may assume that on $\mathbb{D}$ the difference between elements of $\{\lambda, \lambda_3, \dots, \lambda_n\}$ is bounded below by ε_λ.

By rescaling the Riemannian metric on g_C, we may assume that each disk $\mathbb{D}_p$ centered at p has radius at least one. Consequently, assume $\mathbb{D}$ is the disk of radius one.

1.3 Candidate limiting configurations

Definition 1.3.1. We call a triple $(\bar\partial_E, \varphi, h_\infty)$ on $E \to C$ **a** $SU(n)$**-candidate limiting configuration** if

1. $\bar\partial_E\varphi = 0$,

it is an $SU(n)$ configuration in the sense that

2. the triple $\left(\bar\partial_{\mathrm{Det}E}, \mathrm{Tr}\ \varphi, \mathrm{Det}(h_\infty)\right)$ of induced data on the determinant line bundle $\mathrm{Det}E$ is fixed.

and a limiting configuration in the sense that

3. $[\varphi, \varphi^{\dagger_{h_\infty}}] = 0$

4. the Chern connection D_∞ on E associated to the pair $(\bar\partial_E, h_\infty)$ satisfies

$$F_{D_\infty} = -\sqrt{-1}\frac{\deg\ E}{\mathrm{rank}\ E}\mathrm{Id}_E\frac{2\pi\omega}{\mathrm{vol}_g(C)} \tag{1.15}$$

A $SU(n)$ - *candidate* limiting configuration is a $SU(n)$**- limiting configuration** if in addition,

5. The (singular) hermitian metric h_∞ is the $C^\infty_{\mathrm{loc}}(C-Z)$-limit of a family $(\bar\partial_E, \varphi, h_t)$ of solutions of the rescaled Hitchin's equations.

Remark 1.3.1. Requirements (3) and (4) are appropriate conditions because of the "asymptotic decoupling" of Hitchin's equations, exhibited in the following theorem of Mochizuki:

Theorem (2.7 [Moc15]) *On a compact subset $\overline{U}$ of $C-Z$, there exist positive constants c_0 and ε_0 such that if $(\bar\partial_E, \varphi, h_t)$ is a solution of the rescaled Hitchin's equations then, at any point in $\overline{U}$*

$$\left|[\varphi, \varphi^{\dagger_{h_t}}]\right|_{h_t, g_C} \leq c_0 \exp(-\varepsilon_0 t). \tag{1.16}$$

Note that $[\varphi, \varphi^{\dagger_{h_t}}] \in \Omega^{1,1}(C, \mathrm{End}(E))$, so the norm involves both the metric h_t on E and the metric g_C on C, as indicated.

1.3.1 Construction of Candidate Limiting Configurations

In this section, we construct candidate limiting configurations. Looking forward, in Corollary 1.5.2, we prove that the candidate limiting configuration we construct is an *bona fide* limiting configuration.

Proposition 1.3.1. *Let $(\bar\partial_E, \varphi)$ be a simple stable Higgs bundle. Then there is a singular hermitian metric h_∞ such that $(\bar\partial_E, \varphi, h_\infty)$ is a candidate limiting configuration.*

In Construction 1.3.2, we construct a Hermitian metric h_∞ on E. The proof of Proposition 1.3.1 follows. We show that $(\bar\partial_E, \varphi, h_\infty)$ is a candidate limiting configuration.

Construction 1.3.2. *Let* $(\bar{\partial}_E, \varphi)$ *be a simple stable Higgs bundle with associated spectral data* $\mathcal{L} \to \Sigma$.

- *Equip the holomorphic line bundle* $\mathcal{L} \to \Sigma$ *with parabolic structure:*

  Given a ramification point $p \in Z \subset C$, let $\tilde{p} \in \Sigma$ be the point at which the spectral cover is ramified, as shown in Figure 1.2. Call the collection of all such

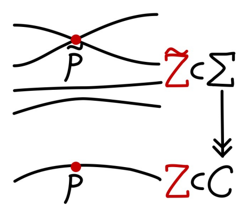

Figure 1.2: $\tilde{Z}$ *is the set of ramification points on the spectral cover* Σ.

points $\tilde{Z}$. Put parabolic weights $\alpha_{\tilde{p}} = -\frac{1}{2}$ at each of the points of $\tilde{p} \in \tilde{Z} \subset \Sigma$. (In the proof of Proposition 1.3.1, the choice of parabolic weights $-\frac{1}{2}$ is critical for $F_{D_\infty} = -\sqrt{-1}\frac{\deg E}{\mathrm{rank} E}\mathrm{Id}_E \frac{2\pi\omega}{\mathrm{vol}_g(C)}$.) With these weights, $\mathcal{L} \to \Sigma$ is a parabolic line bundle with parabolic degree equal to the degree of E, as the following computation shows:

$$\begin{aligned}
\mathrm{pdeg}\, \mathcal{L} &= \deg \mathcal{L} + \sum_{\tilde{p} \in \tilde{Z}} \alpha_p \qquad (1.17)\\
&= \big(\deg E + (n^2 - n)(g-1)\big) + 2(n^2 - n)(g-1)(-\frac{1}{2})\\
&= \deg E.
\end{aligned}$$

Note that we use Grothendieck-Riemann-Roch and the fact that $|\tilde{Z}| = |Z| = 2(n^2 - n)(g-1)$.

- *Equip the parabolic line bundle $\mathcal{L} \to \Sigma$ with a Hermitian structure:*

  For parabolic line bundles—such as $\mathcal{L}$— there is a Hermitian-Einstein metric adapted to the parabolic structure[Sim90, Biq96]. The Hermitian-Einstein condition for $h_{\mathcal{L}}$ is

$$F_{\mathcal{L}} = -\sqrt{-1}\frac{\text{pdeg } \mathcal{L}}{\text{rank } \mathcal{L}}\text{Id}_{\mathcal{L}}\frac{2\pi\pi^*\omega}{\text{vol}_{\pi^*g}(\Sigma)}. \tag{1.18}$$

  This Hermitian-Einstein metric is unique up to a constant.

- *Define h_∞ on $E|_{C-Z}$ from the orthogonal push-forward of the Hermitian-Einstein metric $h_{\mathcal{L}}$ on $\mathcal{L} \to \Sigma$. I.e. decompose E into eigenspaces of φ; these eigenspaces are orthogonal; on each eigenspace, take the metric induced by $h_{\mathcal{L}}$.*

Proof of Proposition 1.3.1. Construction 1.3.2 produced a Hermitian metric h_∞ on E, defined up to a constant. We claim that we can rescale h_∞ in such a way that $(\bar\partial_E, \varphi, h_\infty)$ is a candidate $SL(n, \mathbb{C})$ limiting configuration.

Proof that $\bar\partial_E \varphi = 0$: Because $(\bar\partial_E, \varphi)$ comes as a Higgs bundle, this condition is satisfied.

Proof that $(\bar\partial_E, \varphi, h_\infty)$ is an $SL(n, \mathbb{C})$-solution: The metric h_∞ on E induces a Hermitian-Einstein metric $\text{Det}(h_\infty)$ on Det E. Because Hermitian-Einstein metrics are unique up to a constant, $\text{Det}(h_\infty)$ is a constant multiple of $h_{\text{Det } E}$. Rescale h_∞ by a constant so that $\text{Det}(h_\infty)$ is $h_{\text{Det } E}$.

Proof that $\left[\varphi, \varphi^{\dagger_{h_\infty}}\right] = 0$: Both φ and h_∞ are diagonal in the basis of eigenbundles of φ, hence the commutator vanishes.

Proof that $F_{D_\infty} = -\sqrt{-1}\frac{\deg E}{\text{rank } E}\text{Id}_E\frac{2\pi\omega}{\text{vol}_g(C)}$: This is true precisely because we chose the parabolic weights so that pdeg $\mathcal{L} = \deg E$. We compute

$$\begin{aligned} F_{D_\infty} &= \pi_* F_{\mathcal{L}} \\ &\overset{Eq.\ 1.18}{=} \pi_*\left(-\sqrt{-1}\frac{\text{pdeg } \mathcal{L}}{\text{rank } \mathcal{L}}\text{Id}_{\mathcal{L}}\frac{2\pi\pi^*\omega}{\text{vol}_{\pi^* g}(\Sigma)}\right) \\ &= -\sqrt{-1}\left(\text{pdeg } \mathcal{L}\right)\text{Id}_E\frac{2\pi\omega}{n\text{vol}_g(C)} \\ &\overset{\text{pdeg } \mathcal{L}=\deg E}{=} -\sqrt{-1}\frac{\deg E}{n}\text{Id}_E\frac{2\pi\omega}{\text{vol}_g(C)} \end{aligned} \tag{1.19}$$

□

Proposition 1.3.3. *Let p be in $Z \subset C$. In the holomorphic gauge in Proposition 1.2.2, the candidate (singular) limiting Hermitian metric h_∞ takes the form*

$$h_\infty = \begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \mu_3 & & \\ & & & \cdots & \\ & & & & \mu_n \end{pmatrix} \cdot \begin{pmatrix} |z|^{1/2} & & & & \\ & |z|^{-1/2} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{1.20}$$

for real-valued functions μ, μ_i solving $\bar{\partial}\partial \log \mu_\bullet = 0$.

Notation. The functions μ, μ_i in Proposition 1.3.3 will be appear in the other Hermitian metrics on $E \to C$ locally near p.

Proof. Let $h_{\mathcal{L}}$ be the Hermitian-Einstein metric on $\mathcal{L} \to \Sigma$ which is adapted to the hermitian metric. Let $\{s_i\}$ be the basis of $\pi^*E \to \Sigma$ described in the proof of

Proposition 1.2.2, particularly Eq. 1.11. Then $\pi^* h_\infty$ is

$$\pi^* h_\infty = \begin{pmatrix} |w|^{-1}\nu & & & & \\ & |w|^{-1}\sigma^*\nu & & & \\ & & \nu_3 & & \\ & & & \ddots & \\ & & & & \nu_n \end{pmatrix} \tag{1.21}$$

for some functions $\nu_\bullet$. Because $h_{\mathcal{L}}$ is adapted to the parabolic structure on $\mathcal{L} \to \Sigma$ at $\tilde{p}$, $h_{\mathcal{L}} \sim |w|^{2\alpha_p} = |w|^{2\left(-\frac{1}{2}\right)} = |w|^{-1}$ near $\tilde{p}$. Then in the basis e_i of $E \to C$

$$h_\infty = \begin{pmatrix} |w|^{-1}|w|^2(\nu + \sigma^*\nu) & & & & \\ & |w|^{-1}(\nu + \sigma^*\nu) & & & \\ & & \nu_3 & & \\ & & & \ddots & \\ & & & & \nu_n \end{pmatrix}. \tag{1.22}$$

For $i = 3, \cdots, n$, let $\mu_i = \nu_i$. Let $\mu = \nu + \sigma^*\nu$. Then, because h_∞ satisfies the decoupled Hitchin's equations (Eq. 1.15), $\bar{\partial}\partial \log \mu_\bullet = 0$. $\square$

1.4 A family of approximate solutions

In this section, given a simple stable Higgs bundle $(\bar{\partial}_E, \varphi)$ we construct a family of candidate approximate solutions h_t^{app} by desingularizing the candidate limiting configuration h_∞. The metric h_t^{app} is built from two ingredients: h_∞ and a family of "fiducial solutions" h_t^{fid}.

The metric h_∞ is singular at $p \in Z$, shown in Figure 1.3 by blue spikes. As shown in Figure 1.3, we desingularize h_∞ (shown in blue) by gluing in smooth solutions (shown in orange) of Hitchin's equations on the disks $\mathbb{D}$ around each point $p \in Z$.

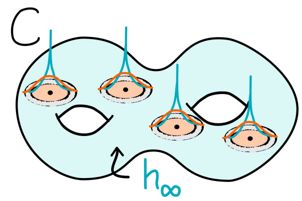

Figure 1.3: *Approximate solutions are constructed by desingularizing h_∞.*

From Proposition 1.2.2 and 1.3.3, there is a neighborhood $\mathbb{D}$ around $p \in Z$ with local holomorphic coordinate centered at p in which

$$
\begin{aligned}
\bar{\partial}_E &= \bar{\partial} \\
\varphi &= \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda_3 & & \\ & & & \cdots & \\ & & & & \lambda_n \end{pmatrix} dz + \begin{pmatrix} 0 & 1 & & & \\ z & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} dz \\
h_\infty &= \begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \mu_3 & & \\ & & & \cdots & \\ & & & & \mu_n \end{pmatrix} \cdot \begin{pmatrix} |z|^{1/2} & & & & \\ & |z|^{-1/2} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}
\end{aligned} \tag{1.23}
$$

We will do the gluing in this gauge.

1.4.1 Fiducial Solution

Note that h_∞ is singular only in the first two entries. Consequently, we use the same smoothing model to desingularize h_∞ for $SL(n, \mathbb{C})$ that Mazzeo-Swoboda-Weiss-Witt used for $SL(2, \mathbb{C})$. This model solution is frequently called the "fiducial solution."

Definition 1.4.1. The $SL(2,\mathbb{C})$ t**-fiducial solution** is

$$\begin{aligned}\bar\partial_E^{(2)} &= \bar\partial \\ \varphi^{(2)} &= \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz \\ h_t^{(2),\mathrm{fid}} &= \begin{pmatrix} |z|^{1/2} e^{u_t(|z|)} & \\ & |z|^{-1/2} e^{-u_t(|z|)} \end{pmatrix}\end{aligned} \tag{1.24}$$

where $u_t : \mathbb{R}^{>0} \to \mathbb{R}$ is solution of

$$\left(\frac{\partial^2}{\partial |z|^2} + \frac{1}{|z|} \frac{\partial}{\partial |z|} \right) u_t = 8t^2 |z| \sinh(2u_t). \tag{1.25}$$

with asymptotics

$$\begin{aligned} u_t(|z|) &\sim \tfrac{1}{\pi} K_0\!\left(\tfrac{8t}{3} |z|^{\frac{3}{2}}\right) && \text{as } |z| \to \infty \\ u_t(|z|) &\sim -\log\left(\tfrac{8t}{3\mathrm{e}^{-2}}\right) \log(|z|) + \tfrac{1}{2} \tfrac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} && \text{as } |z| \to 0. \end{aligned}$$

Remark 1.4.1. In unitary gauge, the fiducial solution is written

$$\begin{aligned} A_{\mathrm{un}}^{(2)} = &\quad d + h^{\frac{1}{2}} \bar\partial h^{-\frac{1}{2}} + h^{-\frac{1}{2}} \partial h^{\frac{1}{2}} &&= d + \left(\frac{1}{8} + \frac{|z|}{4} \frac{du_t}{d|z|} \right) \left(\frac{dz}{z} - \frac{d\bar z}{\bar z} \right) \\ \varphi_{\mathrm{un}}^{(2)} = &\quad h^{\frac{1}{2}} \varphi h^{-\frac{1}{2}} &&= \begin{pmatrix} 0 & |z|^{1/2} e^{u_t(|z|)} \\ \frac{z}{|z|^{1/2}} e^{-u_t(|z|)} & 0 \end{pmatrix} dz \end{aligned} \tag{1.26}$$

In [MSWW14], as well as in [GMN09], the fiducial solution takes this shape.

Notation. Mazzeo-Swoboda-Weiss-Witt call the function u_t solving Eq. 1.25 "h_t" instead. For us, h_t is family of harmonic metrics on E.

1.4.2 Approximate solutions

In this section, we define the family h_t^{app} of candidate approximate solutions, built by desingularizing h_∞ using $h_t^{(2),\mathrm{fid}}$. We define the following non-linear operator to measure the failure of $(\bar\partial_E, \varphi, h_t^{\mathrm{app}})$ to be a solution of Hitchin's

$$\mathbf{F}_t(\bar\partial_E, \varphi, H) := h^{1/2} \left(F_{D(\bar\partial_E, h)} + t^2 [\varphi, \varphi^{\dagger_h}] \right) h^{-1/2}. \tag{1.27}$$

Notation. Let $C^{\text{int}} = \cup_p \mathbb{D}_p$ and $C^{\text{ext}} = C - \overline{C^{\text{int}}}$.

Definition/Proposition 1.4.1. *Choose a smooth radially-symmetric cut-off function $\chi : \mathbb{D} \to [0,1]$ such that*

$$\chi\Big|_{\mathbb{D}_{1/2}} = 1 \quad \textit{and} \quad \operatorname{supp} \chi \subset \mathbb{D}. \tag{1.28}$$

On the disk $\mathbb{D}_p$ centered at $p \in Z$, define h_t^{app} by

$$h_t^{\text{app}} \;=\; \begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \mu_3 & & \\ & & & \cdots & \\ & & & & \mu_n \end{pmatrix} \cdot \begin{pmatrix} |z|^{-1/2} e^{-u_t\chi} & & & & \\ & |z|^{1/2} e^{u_t\chi} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{1.29}$$

On C^{ext}, define $h_t^{\text{app}} = h_\infty$.

For $t_0 > 0$ sufficiently large, there exists positive constants c, δ such that for $t > t_0$

$$\left\| \mathbf{F}_t(\bar{\partial}_E, \varphi, h_t^{\text{app}}) \right\|_{L^2(C)} \le c e^{-\delta t}. \tag{1.30}$$

Because of the exponential decay in t, call the family $\{h_t^{\text{app}}\}_{t>0}$ a ***family of candidate approximate solutions*** *of the rescaled Hitchin's equations.*

Remark 1.4.2. While we've shown that the family h_t^{app} is close to solving Hitchin's equations when t is large, we have not shown that the family h_t^{app} is close to the family of harmonic metric h_t which solve Hitchin's equations. This is the content of the main theorem, Theorem 1.5.1. Until then, we call h_t^{app} a family of *candidate* approximate solutions.

Notation. In the proof of Proposition 1.4.1 and elsewhere, it will be convenient to decompose φ and h_t^{app} on the neighborhood $\mathbb{D} \ni p$ into the following pieces:

$$\begin{aligned}
\varphi^{(\mathrm{t})} &= \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda_3 & & \\ & & & \cdots & \\ & & & & \lambda_n \end{pmatrix} dz && (1.31)\\
\varphi^{(2)} \oplus \mathbf{0}_{n-2} &= \begin{pmatrix} 0 & 1 & & & \\ z & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} dz \\
h^{(\mathrm{t})} &= \begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \mu_3 & & \\ & & & \cdots & \\ & & & & \mu_n \end{pmatrix} \\
h_t^{(2),\mathrm{app}} \oplus \mathbf{1}_{n-2} &= \begin{pmatrix} |z|^{-1/2}e^{-u_t\chi} & & & & \\ & |z|^{1/2}e^{u_t\chi} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.
\end{aligned}$$

Then, on $\mathbb{D}$,

$$\begin{aligned}
\varphi &= \varphi^{(\mathrm{t})} + \left(\varphi^{(2)} \oplus \mathbf{0}_{n-2}\right) && (1.32)\\
h_t^{\mathrm{app}} &= h^{(\mathrm{t})} \cdot \left(h_t^{(2),\mathrm{app}} \oplus \mathbf{1}_{n-2}\right)
\end{aligned}$$

Proof of Proposition 1.4.1. On C^{ext}, $h_t^{\mathrm{app}} = h_\infty$. Because $(\bar\partial_E, \varphi, h_\infty)$ is a candidate solution of Hitchin's equations—and hence solves the decoupled Hitchin's equations—, $\mathbf{F}_t(\bar\partial_E, \varphi, h_t^{\mathrm{app}})$ vanishes on C^{ext}.

On $\mathbb{D}_p$, we have the decomposition of φ and h_t^{app} in Eq. 1.32. Because elements of the pairs $\left(\varphi^{(\mathrm{t})}, h^{(\mathrm{t})}\right)$ and $\left(\varphi^{(2)} \oplus \mathbf{0}_{n-2}, h_t^{(2),\mathrm{app}} \oplus \mathbf{1}_{n-2}\right)$ commute with all other elements of the opposite pairs, Hitchin's equations also split:

$$\begin{aligned} F_{D(\bar{\partial}_E, h_t^{\mathrm{app}})} + t^2[\varphi, \varphi^{\dagger_{h_t^{\mathrm{app}}}}] \quad &= \quad \left(F_{D(\bar{\partial}, h^{(\mathrm{t})})} + t^2\left[\varphi^{(\mathrm{t})}, \left(\varphi^{(\mathrm{t})}\right)^{\dagger_{h^{(\mathrm{t})}}}\right]\right) \\ &+ \left(F_{D(\bar{\partial}^{(2)}, h_t^{(2),\mathrm{app}})} + t^2\left[\varphi^{(2)}, \left(\varphi^{(2)}\right)^{\dagger_{h_t^{(2),\mathrm{app}}}}\right]\right) \oplus \mathbf{0}_{n-2} \end{aligned} \tag{1.33}$$

The triple $(\bar{\partial}_E, \varphi^{(\mathrm{t})}, H^{(\mathrm{t})})$ is is a solution of Hitchin's equations, hence

$$\left\|\mathbf{F}_t(\bar{\partial}, \varphi, h_t^{\mathrm{app}})\right\|_{L^2(\mathbb{D}_p)} = \left\|\mathbf{F}_t(\bar{\partial}^{(2)}, \varphi^{(2)}, h_t^{(2),\mathrm{app}})\right\|_{L^2(\mathbb{D}_p)}. \tag{1.34}$$

By Lemma 6.2 of [MSWW14], for $t_0 > 0$ sufficiently large, there exist positive constants c', δ such that for $t > t_0$

$$\left\|\mathbf{F}_t(\bar{\partial}^{(2)}, \varphi^{(2)}, h_t^{(2),\mathrm{app}})\right\|_{L^2(\mathbb{D}_p)} \leq c' e^{-\delta t}. \tag{1.35}$$

Consequently, for $t > t_0$, the same exponent δ, and constant $c = \sqrt{|Z|}c'$,

$$\left\|\mathbf{F}_t(\bar{\partial}_E, \varphi, h_t^{\mathrm{app}})\right\|_{L^2(C)} \leq c e^{-\delta t}. \tag{1.36}$$

□

Remark 1.4.3. Note that h_t^{app} fails to be a solution of Hitchin's equations only on the gluing annuli $\mathbb{D} - \overline{\mathbb{D}_{1/2}}$ around each ramification point $p \in Z \subset C$.

1.5 Perturbing to a solution of Hitchin's equations

In Eq. 1.27, we defined a non-linear operator

$$\mathbf{F}_t(\bar{\partial}_E, \varphi, h) := h^{1/2}\left(F_{D(\bar{\partial}_E, h)} + t^2[\varphi, \varphi^{\dagger_h}]\right) h^{-1/2}.$$

The output is the failure of $(\bar{\partial}_E, \varphi, h)$ to be a solution of the t-rescaled Hitchin's equations. Since we fix the underlying Higgs bundle $(\bar{\partial}_E, \varphi)$, we view $\mathbf{F}_t$ as an operator on Hermitian metrics.

In Proposition 1.4.1, we proved that the family h_t^{app} was close to solving Hitchin's equations. The upcoming main theorem (Theorem 1.5.1) states says something much stronger: for $t \gg 0$, the approximate metric h_t^{app} is close to the actual harmonic h_t solving Hitchin's equations.

To state the theorem, we express the family of harmonic metrics, h_t, in terms of the family of approximate metrics, h_t^{app}. To match the conventions of Mazzeo-Swoboda-Weiss-Witt, we first switch to unitary gauge.

$$\begin{aligned} \Phi_t &= (h_t^{\mathrm{app}})^{1/2} \circ \varphi \circ (h_t^{\mathrm{app}})^{-1/2} \\ A_t^{0,1} &= (h_t^{\mathrm{app}})^{1/2} \circ \bar{\partial}_E \circ (h_t^{\mathrm{app}})^{-1/2} \end{aligned} \tag{1.37}$$

A complex gauge transformation g maps the triple $(\Phi_t, A_t^{0,1}, \mathrm{Id})$ to

$$(\Phi_t, A_t^{0,1}, \mathrm{Id})^g = (g^{-1}\Phi_t g, g^{-1} \circ A_t^{0,1} \circ g, (gg^*)^{-1}). \tag{1.38}$$

Hitchin's equations are invariant under unitary gauge transformations. Consequently, we assume that the gauge transformation g is Hermitian, taking the standard slice of the complex gauge transformations modulo unitary gauge transformations. Let $g = \mathrm{e}^{\gamma}$. We redefine the operator $\mathbf{F}_t$ to be

$$\mathbf{F}_t^{\mathrm{app}}(\gamma) := -i \star \left(F_{A_t^{\exp(\gamma)}} + t^2 [\mathrm{e}^{-\gamma}\Phi_t \mathrm{e}^{\gamma}, \mathrm{e}^{\gamma}\Phi_t^* \mathrm{e}^{-\gamma}] \right). \tag{1.39}$$

We add the superscript to clarify that this non-linear expression is based at h_t^{app}, even if the relationship between γ, h_t^{app}, and the harmonic metric h_t has been obscured.

Main Theorem 1.5.1. *There is a value $m > 0$, such that for t sufficiently large, there is a unique Hermitian γ_t satisfying $\|\gamma_t\|_{H^2(i\mathfrak{su}(E))} \leq t^{-m}$ such $\mathbf{F}_t^{\mathrm{app}}(\gamma) = 0$, i.e. $(A_t^{\exp(\gamma_t)}, \Phi_t^{\exp(\gamma_t})})$ solves Hitchin's equations.*

(The next subsection gives a proof of the Theorem 1.5.1. The proof of Theorem 1.5.1 requires a number of results. For readability, the proofs of these results are delayed until §1.6.)

Because the family h_t is close to h_t^{app}, and the family h_t^{app} converges to h_∞, the family h_t also converges to h_∞ in the following sense:

Corollary 1.5.2. *h_t converges to h_∞ in the $C^\infty_{\mathrm{loc}}(C - Z)$. Consequently, the candidate limiting configuration h_∞ is an actual limiting configuration, in the sense of Definition 1.3.1.*

1.5.1 Proof of Theorem

Theorem 1.5.1 is proved using a contraction mapping argument, as in Mazzeo-Swoboda-Weiss-Witt [MSWW14].

In Eq. 1.39, we defined $\mathbf{F}_t^{\mathrm{app}}(\gamma)$ acting on a hermitian section γ. The operator $\mathbf{F}_t^{\mathrm{app}}$ is naturally a map between the following Sobolev spaces

$$\mathbf{F}_t^{\mathrm{app}} : H^2(i\mathfrak{su}(E)) \quad \rightarrow \quad L^2(i\mathfrak{su}(E)) \tag{1.40}$$

where $H^2 = W^{2,2}$, for convenience.

We construct γ_t using a contraction mapping argument. Observe that $\mathbf{F}_t^{\mathrm{app}}(\gamma_t) = 0$ if γ_t is a fixed point of the map

$$\begin{aligned} \mathbf{T}_t : H^2(i\mathfrak{su}(E)) &\to H^2(i\mathfrak{su}(E). \\ \gamma &\mapsto \gamma - \left(D\mathbf{F}_t^{\mathrm{app}}\right)^{-1}\left(\mathbf{F}_t^{\mathrm{app}}(\gamma)\right). \end{aligned} \tag{1.41}$$

This presupposes that the linearization $D\mathbf{F}_t^{\mathrm{app}}$ has an inverse— a fact we prove in Proposition 1.6.1. For convenience, we will abbreviate $D\mathbf{F}_t^{\mathrm{app}}$ by L_t. Expand $\mathbf{F}_t^{\mathrm{app}}$ into constant, linear, and non-linear term:

$$\mathbf{F}_t^{\mathrm{app}}(\gamma) = \mathbf{F}_t^{\mathrm{app}}(0) + L_t(\gamma) + Q_t(\gamma). \tag{1.42}$$

Note then that

$$\mathbf{T}_t(\gamma) = -(L_t)^{-1}(\mathbf{F}_t^{\mathrm{app}}(0) + Q_t(\gamma)). \tag{1.43}$$

To prove that $\mathbf{T}_t$ has a fixed point, we need to show that there is some ball $B_{\rho_t} \in H^2(i\mathfrak{su}(E))$ centered at the zero section (corresponding to h_t^{app}) on which $\mathbf{T}_t$ is a contraction mapping of B_{ρ_t}. We will need the following:

- (Lemma 1.5.3) There is a constant $\hat{C} > 0$ such that

$$\|Q_t(\gamma_1) - Q_t(\gamma_2)\|_{L^2} \leq \hat{C}\rho t^2 \|\gamma_1 - \gamma_2\|_{H^2} \tag{1.44}$$

  for all $\rho \in (0, \varepsilon]$ and $\gamma_0, \gamma_1 \in B_\rho \subset H^2$.

- (Proposition 1.6.2) For $t_0 > 0$ sufficiently large, there is a constant $\tilde{C}$ such that $\|L_t^{-1}\|_{\mathcal{L}(L^2,H^2)} \leq \tilde{C}t^2$ for all $t > t_0$.

The contraction mapping argument and the estimate on Q_t are the same as in Mazzeo-Swoboda-Weiss-Witt. The proof of the estimate on the L_t differs in places from Mazzeo-Swoboda-Weiss-Witt's proof. The proof of Proposition 1.6.2 makes up §1.6.

Proof of Main Theorem 1.5.1. By Lemma 1.6.1, the linearization L_t has an inverse, consequently the operator $\mathbf{T}_t$ defined in Eq. 1.43 is defined. For all $\rho \in (0,1]$ and $t > t_0$, Hermitian sections $\gamma_1, \gamma_2 \in B_\rho$ satisfy

$$\begin{aligned} \|\mathbf{T}_t(\gamma_1 - \gamma_2)\|_{H^2} &= \| - L_t^{-1}\left(Q_t(\gamma_1) - Q_t(\gamma_2)\right)\|_{H^2} \\ &\leq \|L_t^{-1}\|_{\mathcal{L}(L^2,H^2)}\|\|Q_t(\gamma_1) - Q_t(\gamma_2)\|_{L^2} \\ &\overset{\textit{Lem } 1.5.3,\ \textit{Prop } 1.6.2}{\leq} \tilde{C}t^2 \cdot \hat{C}\rho t^2 \|\gamma_1 - \gamma_2\|_{L^2}. \end{aligned} \tag{1.45}$$

Consequently $\mathbf{T}_t$ is a contraction on the ball of radius $r_t = \frac{1}{\tilde{C}\hat{C}t^4}$.

Note that

$$\begin{aligned} \|\mathbf{T}_t(\gamma)\|_{H^2} &\leq \tilde{C}t^2 \cdot \hat{C}\rho t^2 \|\gamma\|_{L^2} + \|\mathbf{T}_t(0)\|_{H^2} \\ &\leq \tilde{C}t^2 \cdot \hat{C}\rho t^2 \|\gamma\|_{L^2} + \tilde{C}t^2 C e^{-\delta t} \end{aligned} \tag{1.46}$$

Because $\|\mathbf{T}_t(0)\|$ exhibits exponential decay in t, for large enough t there eventually a radius ρ_t for which $\mathbf{T}_t(B_{\rho_t}) \subset B_{\rho_t}$. Consequently, there is a a unique fixed point of $\mathbf{T}_t$. This is γ_t. $\square$

1.5.2 Estimates on non-linear term, Q_t

The nonlinear term Q_t is given in Eq. 1.42. For $SL(2,\mathbb{C})$, Mazzeo-Swoboda-Weiss-Witt prove

Lemma([MSWW14], Lemma 6.9) *There exists a constant $C > 0$ such that*

$$\|Q_t(\gamma_1) - Q_t(\gamma_2)\|_{L^2} \leq C\rho t^2 \|\gamma_1 - \gamma_2\|_{H^2} \tag{1.47}$$

for all $\rho \in (0,1]$ and $\gamma_0, \gamma_1 \in B_\rho \subset H^2(i\mathfrak{su}(E))$.

Their proof does not rely on the rank of E. It just relies on some estimates on A_t given in [MSWW14], Lemma 6.8. In our case, the same estimate holds, consequently, we get the $SL(n,\mathbb{C})$ analog of Mazzeo-Swoboda-Weiss-Witt's lemma here.

Lemma 1.5.3. *There exists a constant $\hat{C} > 0$ such that*

$$\|Q_t(\gamma_1) - Q_t(\gamma_2)\|_{L^2} \leq \hat{C}\rho t^2 \|\gamma_1 - \gamma_2\|_{H^2} \tag{1.48}$$

for all $\rho \in (0,\varepsilon]$ and $\gamma_0, \gamma_1 \in B_\rho \subset H^2$.

Remark 1.5.1. For a very rough idea of what kind of estimate this is, take $\gamma \in \mathbb{R}$, $Q_t(\gamma) = t^2\gamma^m$. For which m does Q_t satisfy the above lemma? Taking $\gamma_1 = \rho, \gamma_2 = 0$, see that m must satisfy

$$\|Q_t(\rho)\| = t^2\rho^m \leq C\rho t^2 \rho, \tag{1.49}$$

for all $\rho \in (0,1]$. Consequently, see that $m \geq 2$, i.e. the nonlinear terms are at least quadratic.

1.6 Properties of the linearization

The linearization of $\mathbf{F}_t^{\mathrm{app}}$ (Eq. 1.39) at 0 is

$$D\mathbf{F}_t^{\mathrm{app}}(0)[\gamma] \quad := \quad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{F}_t^{\mathrm{app}}((\varepsilon t)^2) \tag{1.50}$$

$$= \quad \Delta_{A_t}\gamma - i \star t^2 M_{\Phi_t}\gamma$$

where

$$\begin{aligned} \Delta_{A_t}\gamma &:= d_{A_t}^* d_{A_t}\gamma \\ M_{\Phi_t}\gamma &:= [\Phi_t^* \wedge [\Phi_t, \gamma]] - [\Phi_t \wedge [\Phi_t^*, \gamma]] \,. \end{aligned} \tag{1.51}$$

For convenience, we abbreviate the linearization by L_t.

$$L_t(\gamma) = D\mathbf{F}_t^{\text{app}}(0)[\gamma] = \Delta_{A_t}\gamma - i \star t^2 M_{\Phi_t}\gamma. \tag{1.52}$$

Note that $L_t : L^2 \to L^2$ is self-adjoint. In this section we prove two propositions. First, we prove that there is a lower bound on the first eigenvalue of L_t, which is uniform in t.

Proposition 1.6.1. *The linear operator L_t has an inverse. Moreover, for t_0 sufficiently large, there is a constant $C > 0$ such that*

$$\|L_t\gamma\|_{L^2} \geq C\|\gamma\|_{L^2}. \tag{1.53}$$

for $t > t_0$.

The following proposition builds on the above proposition and is required for the proof of Theorem 1.5.1.

Proposition 1.6.2. *For t_0 sufficiently large, there is a constant $\tilde{C} > 0$ such that*

$$\|L_t^{-1}\|_{\mathcal{L}(L^2,H^2)} \leq \tilde{C}t^2. \tag{1.54}$$

if $t > t_0$.

The next section (§1.6.1) consists of the proof of Proposition 1.6.1. The proof of Proposition 1.6.2 follows in §1.6.2.

1.6.1 Proof of Proposition 1.6.1

For the $SL(2,\mathbb{C})$ case, the analog of Proposition 1.6.1 is stated by Mazzeo-Swoboda-Weiss-Witt in [MSWW14] Lemma 6.3. An important ingredient of their strategy is the the domain decomposition principle, in which they decompose C into disjoint pieces: neighborhoods $\mathbb{D}_p$ around each $p \in Z$ and the remaining piece $C^{\text{ext}} = C - \cup_p \mathbb{D}_p$. On each piece, they prove that the first Neumann eigenvalue of L_t if bounded below by some positive constant. The domain decomposition principle gives a positive first global eigenvalue of L_t.

One might hope that the proof for $SL(n,\mathbb{C})$ is exactly the same for $n > 2$. However, this does not work because the Neumann boundary problem on each disk $\mathbb{D}$ has kernel. By explicit computation of L_t in the basis Proposition 1.2.2, 1.3.3 on $\mathbb{D}$ one can compute that this kernel consists of constant traceless diagonal matrices with the shape

$$\gamma = \begin{pmatrix} \alpha & & & & \\ & \alpha & & & \\ & & \alpha_3 & & \\ & & & \ddots & \\ & & & & \alpha_n \end{pmatrix}. \tag{1.55}$$

However, the simplicity of the Neumann kernel suggests that this kernel is an artifact of a bad decomposition. Consequently, we pursue a different strategy.

We define an operator

$$\widetilde{L}_t = \Delta_{A_\infty} - i \star t^2 M_{\Phi_\infty} \tag{1.56}$$

where Δ_{A_∞} and M_Φ are defined in Eq. 1.51. (Recall that L_t is

$$L_t(\gamma) = \Delta_{A_t}\gamma - i \star t^2 M_{\Phi_t}\gamma. \tag{1.57}$$

Consequently, the linearization $\widetilde{L}_t$ is computed from the abelian data $(\bar{\partial}_E, \varphi, h_\infty)$ while L_t is computed at the approximate solution $(\bar{\partial}_E, \varphi, h_t^{\text{app}})$.)

This operator is very close to L_t, as the following lemma states.

Lemma 1.6.3. *For t_0 sufficiently large, there are positive constants $\tilde{c}, \delta > 0$ such that*

$$\|\widetilde{L}_t - L_t\|_{L^2} \leq \tilde{c}\mathrm{e}^{-\delta t} \tag{1.58}$$

for $t > t_0$.

Proof. In the definition of h_t^{app} 1.4.1, we see that h_t^{app} differs from h_∞ only the disks $\mathbb{D}$ around each ramification point $p \in Z \subset C$. Moreover, on $E|_{\mathbb{D}}$, h_t^{app} differs from h_∞ only the top left 2×2 block, where we use inserted the t-fiducial solution. (See Eq. 1.31.) Consequently, our analysis is no different from Mazzeo-Swoboda-Weiss-Witt's analysis. They state the analog of Lemma 1.6.3 at the bottom of p. 36. It depends on the exponential decay of the Painlevé III solution $u_t(r)$ for large values of r. □

An advantage of $\widetilde{L}_t$ is that it came from purely abelian data so it decomposes and is particularly easy to analyze.

Lemma 1.6.4. *On $\pi^*\mathrm{End}\, E \to \Sigma$, $\pi^*\widetilde{L}_t$ respects the decomposition*

$$\pi^*\mathrm{End}\, E = \oplus_{i,j}\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j), \tag{1.59}$$

where $\mathcal{L}_i \to \Sigma$ is the eigenlinebundle of φ corresponding to eigenvalue $\pi^*\lambda_i$ on Σ.

Proof. Since h_∞ and φ on Σ come from pushing forward abelian data on $\mathcal{L} \to \Sigma$, the proof is immediate. □

The following facts will be useful, so we collect them here.

Lemma 1.6.5.

- *(cf. [MSWW14] Proposition 5.1)*

$$\left\langle \tilde{L}_t\gamma, \gamma \right\rangle_{L^2} = \|d_{A_\infty}\gamma\|_{L^2}^2 + 4t^2\|[\Phi_\infty, \gamma]\|_{L^2}^2 \tag{1.60}$$

- *There is a constant c_M such that at any point of C*

$$|M_{\Phi_\infty}|_{g_C, h_\infty} \leq c_M. \tag{1.61}$$

- *$\tilde{L}_t$ has no kernel*

Proof. To see the bound on M_{Φ_∞}, note that on $\pi^*\mathrm{End}\, E = \oplus\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j)$, the (i,j) entry of M_{Φ_∞} is

$$(\pi^* M_{\Phi_\infty}\gamma)_{ij} = \left(2|\lambda_i - \lambda_j|^2 dz \wedge d\bar{z}\right)\gamma_{ij}. \tag{1.62}$$

(Here we use that φ and h_∞ commute, hence $\Phi_\infty = \varphi$. In addition, φ diagonalizes over the spectral cover in the basis described.) The difference between the eigenvalues are bounded above, hence $|M_{\Phi_\infty}|$ is bounded above.

To see that $\tilde{L}_t$ has no kernel for $t > 0$, we show that if γ satisfies $[\Phi_\infty, \gamma] = 0$, then $d_{A_\infty}\gamma$ vanishes only if γ is identically zero. This is most apparent again on the spectral cover. Because $[\Phi_\infty, \gamma] = 0$, $\pi^*\gamma$ is diagonal in the decomposition $\pi^*\mathrm{End}\, E = \oplus\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j)$. On the $\pi^*\gamma$ diagonal, $\pi^* d_{A_\infty}$ acts as $\pi^* d$. Consequently, if $d_{A_\infty}\gamma = 0$, then $\pi^*\gamma$ is constant. Because $\pi^*\gamma$ is pulled back from γ, if λ_i is ramified at $\tilde{p} \in \Sigma$, then we have the following symmetry of the (i,i) entry of $\pi^*\gamma$:

$(\pi^*\gamma)_{ii}(-w) = -(\pi^*\gamma)_{ii}(w)$. Consequently $\pi^*\gamma \equiv 0$. Hence $\gamma \equiv 0$, i.e. $\tilde{L}_t$ has no kernel. $\square$

Rather than proving directly that the first eigenvalues of L_t are bounded below by some constant, we first prove that the first eigenvalues of the easier operator $\widetilde{L}_t$ are bounded below by some constant, and then relate the eigenvalues of L_t and the simpler operator, $\widetilde{L}_t$.

Lemma 1.6.6. *There is a positive constant $\kappa > 0$ such that $\|\widetilde{L}_t\gamma\|_{L^2} \geq \kappa\|\gamma\|_{L^2}$ for all $t \geq 1$.*

Proof. By Lemma 1.6.5

$$\left\langle \widetilde{L}_t\gamma, \gamma \right\rangle_{L^2} = \|d_{A_\infty}\|_{L^2}^2 + 4t^2\|[\Phi_\infty, \gamma]\|_{L^2}^2. \tag{1.63}$$

Consequently, the first eigenvalue of $\widetilde{L}_t$ is a non-decreasing function of t. By Lemma 1.6.5, if $[\Phi_\infty, \gamma] = 0$, then $d_{A_\infty}\gamma$ cannot vanish. Consequently, $\widetilde{L}_t$ has no kernel for $t > 0$. Let κ be the first eigenvalue of $\widetilde{L_1}$. Because $\widetilde{L}_t \geq \widetilde{L_1}$ for $t \geq 1$, the first eigenvalue of $\widetilde{L}_t$ is greater than κ for $t \geq 1$. $\square$

We may now prove Proposition 1.6.1, i.e. we show that the first eigenvalues of L_t are bounded below by some constant.

Proof of Proposition 1.6.1. For $t_0 > 1$ sufficiently large (as in Lemma 1.6.3), if $t > t_0$,

$$\begin{aligned} \langle L_t\gamma, \gamma\rangle_{L^2} &= \left\langle \widetilde{L}_t\gamma, \gamma \right\rangle_{L^2} + \left\langle \left(L_t - \widetilde{L}_t\right)\gamma, \gamma \right\rangle_{L^2} \\ &\overset{\text{Lem } 1.6.6}{\geq} \kappa\|\gamma\|_{L^2} - \left| \left\langle \left(L_t - \widetilde{L}_t\right)\gamma, \gamma \right\rangle_{L^2} \right| \end{aligned} \tag{1.64}$$

$$
\begin{aligned}
&\geq \quad \kappa\|\gamma\|_{L^2} - \|L_t - \widetilde{L}_t\|_{\mathcal{L}(L^2,L^2)}\|\gamma\|_{L^2}^2 \\
&\overset{\text{Lem 1.6.3}}{\geq} \quad \kappa\|\gamma\|_{L^2} - \tilde{c}\mathrm{e}^{-\delta t}\|\gamma\|_{L^2}^2 \\
&\geq \quad (\kappa - \tilde{c}\mathrm{e}^{-\delta t})\|\gamma\|_{L^2}^2.
\end{aligned}
$$

For t sufficiently large, $\kappa - \tilde{c}\mathrm{e}^{-\delta t}$ is positive. Consequently, for such sufficiently large fixed t_0, the constant $C = \kappa - \tilde{c}\mathrm{e}^{-\delta t_0}$ satisfies the desired condition, i.e. for any $t > t_0$,

$$
\|L_t\gamma\|_{L^2} \geq C\|\gamma\|_{L^2}. \tag{1.65}
$$

Because L_t is self-adjoint with no kernel and first eigenvalue larger that $C > 0$, L_t has a bounded inverse, L_t^{-1}. □

1.6.2 Proof of Proposition 1.6.2

While the $SL(n,\mathbb{C})$ proof of Proposition 1.6.1 differs from Mazzeo-Swoboda-Weiss-Witt's proof, the rest of the proof of Proposition 1.6.2 follows the proof of Mazzeo-Swoboda-Weiss-Witt.

Proof of 1.6.2. Like Mazzeo-Swoboda-Weiss-Witt, we prove that the operator

$$
L_t^{-1} : L^2(i\mathfrak{su}(E)) \to H^2(i\mathfrak{su}(E)) \tag{1.66}
$$

is bounded by using the fact that the graph norm of Δ_{A_∞} is equivalent to the standard Sobolev H^2-norm. (See [MSWW14] Lemma 6.5.) Consequently, we desire to prove that that there is a constant $\tilde{C}'$ such that

$$
\sqrt{\|L_t^{-1}u\|_{L^2}^2 + \|\Delta_{A_\infty}L_t^{-1}u\|_{L^2}^2} \leq \tilde{C}'t^2\|u\|_{L^2}. \tag{1.67}
$$

First note that if t_0 sufficiently large (as in Proposition 1.6.1), then for $t > t_0$, we have

$$\begin{aligned} \|\Delta_{A_\infty} L_t^{-1} u\|_{L^2} &\leq \|L_t L_t^{-1} u\|_{L^2} + \|(\Delta_{A_\infty} - L_t) L_t^{-1} u\|_{L^2} \\ &\overset{\text{Prop 1.6.1}}{\leq} \|u\|_{L^2} + \|(\Delta_{A_\infty} - L_t)\|_{\mathcal{L}(L^2,L^2)} \frac{1}{C} \|u\|_{L^2}. \end{aligned} \tag{1.68}$$

Consequently, we seek to prove such a bound on $\|(\Delta_{A_\infty} - L_t)\|_{\mathcal{L}(L^2,L^2)}$. We do so by relating both to $\widetilde{L}_t = \Delta_{A_\infty} - i \star t^2 M_{\Phi_\infty}$. For $t > t_0$,

$$\begin{aligned} \|(\Delta_{A_\infty} - L_t)\|_{\mathcal{L}(L^2,L^2)} &\leq \|(\Delta_{A_\infty} - \widetilde{L}_t)\|_{\mathcal{L}(L^2,L^2)} + \|(\widetilde{L}_t - L_t)\|_{\mathcal{L}(L^2,L^2)} \\ &\leq t^2 \|M_{\Phi_\infty}\|_{\mathcal{L}(L^2,L^2)} + \|(\widetilde{L}_t - L_t)\|_{\mathcal{L}(L^2,L^2)} \\ &\overset{\text{Lem 1.6.5,1.6.3}}{\leq} c_M t^2 + \tilde{c} e^{-\delta t} \\ &\leq (c_M + \tilde{c}') t^2 \end{aligned} \tag{1.69}$$

Consequently, for $t > t_0$

$$\begin{aligned} \|\Delta_{A_\infty} L_t^{-1} u\|_{L^2} &\leq \|u\|_{L^2} + \|(\Delta_{A_\infty} - L_t)\|_{\mathcal{L}(L^2,L^2)} \frac{1}{C} \|u\|_{L^2} \\ &\leq \left(\frac{1}{t_0^2} + \frac{c_M + \tilde{c}'}{C} \right) t^2 \|u\|_{L^2}. \end{aligned} \tag{1.70}$$

Take

$$\tilde{C}' = \sqrt{\frac{1}{(Ct_0)^2} + \left(\frac{1}{t_0^2} + \frac{c_M + \tilde{c}'}{C} \right)^2}. \tag{1.71}$$

Then returning to Eq. 1.67, for $t > t_0$, indeed

$$\sqrt{\|L_t^{-1} u\|_{L^2}^2 + \|\Delta_{A_\infty} L_t^{-1} u\|_{L^2}^2} \leq \tilde{C}' t^2 \|u\|_{L^2}^2. \tag{1.72}$$

$\square$

Chapter 2

From S^1-fixed points to $\mathcal{W}$-algebra representations

This is joint work with Andrew Neitzke.

2.1 Introduction

Fix a pair (K, N) of positive, coprime integers K, N. Let $E \to \mathbb{CP}^1$ be a complex vector bundle of rank K over $\mathbb{CP}^1$ with a hermitian metric and a trivialization of Det E. Let (A, Φ) be a pair consisting of

- A, a unitary connection on E that is trivial on Det E, and
- Φ, a traceless End E-valued $(1,0)$-form known as the "Higgs field"

that satisfies Hitchin's equations:

$$\begin{aligned} d_A \Phi &= 0 \\ F_A + [\Phi, \Phi^\dagger] &= 0. \end{aligned} \tag{2.1}$$

Here, $d_A : \Omega^i(C, \text{End } E) \to \Omega^{i+1}(C, \text{End } E)$ is the associated exterior derivative, $F_A = d_A^2$ is the curvature of A and $\Phi^\dagger$ is the hermitian adjoint of Φ. For right now, we avoid discussing the requisite additional boundary conditions of the pair (A, Φ) at infinity.

In this paper we study the solutions of Hitchin's equations which are fixed by a particular circle action:

Definition 2.1.1. Fix K and N. Let z be the usual holomorphic coordinate on $\mathbb{C}$. For $\theta \in \mathbb{R}/(2\pi(K+N))\mathbb{Z}$, define an action of $\mathrm{e}^{i\theta}$ on the space of pairs (A, Φ) by

$$\begin{aligned} z &\overset{\rho}{\mapsto} e^{-i\frac{K}{K+N}\theta} z \\ \Phi &\mapsto e^{i\theta}\rho^*\Phi \\ A &\mapsto \rho^* A. \end{aligned} \tag{2.2}$$

Call this action the S^1**-action by** $\mathrm{e}^{i\theta}$**.**

Remark 2.1.1. ▷ Note that θ takes values in $\mathbb{R}/(2\pi(K+N))\mathbb{Z}$, the $(K+N)$-fold cover of the usual circle $\mathbb{R}/2\pi\mathbb{Z}$.◁

Circle actions on the space of solutions of Hitchin's equations are nothing new. In his seminal paper [Hit87], Hitchin defined an S^1-action on the Hitchin moduli space over a compact Kähler curve C:

$$\begin{aligned} \Phi &\mapsto e^{i\theta}\Phi \\ A &\mapsto A. \end{aligned} \tag{2.3}$$

The moment map μ for this S^1-action is a perfect Morse function on the Hitchin moduli space. Consequently, Hitchin was able to compute the Betti numbers of the $SU(2)$-Hitchin moduli space by studying the S^1-fixed point sets and their indices. Since then, many others have studied S^1-actions on Hitchin moduli spaces, extending Hitchin's results to higher rank groups [Got94] and parabolic Higgs bundles [GPGM05]. Hitchin moduli spaces on a Kähler curve C are diffeomorphic to character varieties of the surface group $\pi_1(C)$. Consequently, S^1-actions on Hitchin moduli spaces are now a standard—if unwieldy—tool to compute the topology of character varieties.

For the S^1-action in Definition 2.1.1, the right Hitchin moduli space has not yet been defined. In the ordinary Hitchin moduli space, the Higgs field Φ must be $\bar{\partial}_A$-holomorphic. Consequently, on $\mathbb{C}P^1$, any holomorphic Higgs field must be identically zero. This is not the relevant moduli space.

There are many more Higgs bundles once we allow the Higgs field to be meromorphic. For any good moduli space, we must give a set of marked points (the locations of the poles of Φ) and the behavior of (A, Φ) near each of these marked points. The right Hitchin moduli space will have a single marked point at infinity. To begin to describe the behavior of (A, Φ) near $\infty \in \mathbb{CP}^1$, we look at the behavior of the pairs (A, Φ) which are fixed by the S^1-action (see Definition 2.2.1) because these pairs (A, Φ) should be points in the moduli space. In particular, we look at the eigenvalues of Φ. If (A, Φ) is an S^1-fixed point, then the eigenvalues of the Higgs field Φ are scalar multiples of $z^{\frac{N}{K}}\mathrm{d}z$ (Lemma 2.2.1). In the coordinate $w = \frac{1}{z}$ centered at ∞, the eigenvalues are scalar multiples of $w^{-\left(\frac{N}{K}+2\right)}\mathrm{d}w$ at ∞. The theory of parabolic Higgs bundles (and the corresponding Hitchin moduli spaces) only accounts for eigenvalues with simple poles. Biquard-Boalch developed a non-abelian Hodge correspondence in the case where the eigenvalues of the Higgs field have poles of order greater than one, and called the corresponding Hitchin moduli spaces "wild Hitchin moduli spaces"

[BB02]. However, Biquard-Boalch assume that pair (A, Φ) satisfies some extra conditions, including the assumption that the polar part of the Higgs field is diagonalizable at the singularities of Φ. However, the S^1-fixed points will not have diagonalizable polar parts because the eigenvalues of Φ have fractional powers $w^{-\left(\frac{N}{K}+2\right)}\mathrm{d}w$ at ∞. Consequently, the right Hitchin moduli space for this S^1-action ought to be some more general "twisted wild Hitchin moduli space."

These twisted wild Hitchin moduli spaces ought to fit into a non-abelian Hodge correspondence. There have been a number of recent results about the objects in the non-abelian Hodge correspondence. However, so far a full non-abelian Hodge correspondence between a Higgs bundle moduli space and a character variety—through a twisted wild Hitchin moduli space—does not exist. Mochizuki wrote a lengthy monograph about the relevant objects [Moc10]. (It is very general. Rather than just bundles over Kähler curves, he considers sheaves over higher-dimensional Kähler manifolds.) In it, he did not assume the polar part of the Higgs field was diagonalizable. He proves a correspondence, at the level of objects and morphisms, between "good filtered Higgs bundles" and "wild harmonic bundles" and meromorphic flat connections. However, there is no mention of gauge group, and so the desired moduli spaces do not appear there. More recently, Boalch-Yamakawa [BY15] studied "twisted wild character varieties." These ought to be the right character varieties corresponding to twisted wild Hitchin moduli spaces in a non-abelian Hodge correspondence. Though twisted wild Hitchin moduli spaces do not exist in the literature, this chapter is conjecturally about certain elements of these spaces. To stay out of the realm of conjecture, our results are stated at the level of objects, and we make no substantive mention of gauge transformations.

Though these twisted wild Hitchin moduli spaces do not exist in the literature yet, there already are a number of conjectures about them. In particular, a number of these conjectures seem to be about the twisted wild Hitchin moduli spaces which contain the fixed points of the (K, N)-S^1-action. For convenience, we'll refer to this moduli space as $\mathcal{M}_{K,N}$— even if it is only conjectural. Gorsky-Oblomkov-Rasmussen-Shende [GORS12] conjecture a relation between the cohomology of a certain compactified Jacobian and the Khovanov-Rozansky homology of a (K, N)-torus knot. This compactified Jacobian should be the most degenerate fiber of the Hitchin fibration of $\mathcal{M}_{K,N}$. From preliminary computations, we expect that the cohomology of this Hitchin fiber will localize at the S^1-fixed points. Consequently, we expect that the S^1-action in Definition 2.1.1 is useful in extracting topological information about the corresponding (K, N)-twisted wild Hitchin moduli spaces, regardless of the size of K and N. Note that this is in sharp contrast to the utility of Hitchin's S^1-action (Eq. 2.3), where extracting topological information is much harder for higher rank vector bundles.

The main theorem of the paper is related to a certain physics conjecture of Cordova-Shao [CS15], which relates certain $\mathcal{N} = 2$ $4d$ supersymmetric theories and $\mathcal{W}$-algebras. As background for their conjecture, given any $\mathcal{N} = 2$ $4D$ supersymmetric theory, one can associate a vertex algebra[BLL$^+$13]. A certain subclass of these theories, known as "theories of class S," are related to the Hitchin moduli spaces[GMN09]. For the usual Hitchin moduli space studied by Hitchin and Simpson [Hit87, Sim88], the data of the related theory of class S is a Lie algebra (e.g. $A_{K-1} = \mathfrak{sl}(K, \mathbb{C})$) and a complex curve C. However, with a more general notion of "complex curve," certain theories of class S are related to more general Hitchin moduli spaces. It is believed that twisted wild Hitchin moduli spaces correspond to theories of class S with defects. For wild Hitchin moduli spaces—both standard and twisted—Gaiotto-Moore-Neitzke suggest that the data of the related theory of class S is a Lie algebra and an "irregular curve" C. (See [GMN09] §3.1.7.) The irregular curve is a complex curve with additional singularity data. The additional singularity data governs the the behavior of all objects near the singular points: the behavior of the bundle with flat connection, the behavior of the solution of Hitchin's equations, and the behavior of the Higgs bundle. While one might think that defects in a theory of class S would add difficulty, in many situations defects simplify computations. Let $T[\mathbb{C}P^1_{K,N}, \mathfrak{sl}(K)]$ denote the theory of class S which is related to the conjectural twisted wild Hitchin moduli space $\mathcal{M}_{K,N}$. Cordova-Shao conjecture that the associated vertex algebra is particularly simple in this case.

Conjecture 2.1.1 (Cordova-Shao)**.** *The vertex algebra associated to* $T[\mathbb{C}P^1_{K,N}, \mathfrak{sl}(K)]$ *is the* $(K, N+K)$ $\mathcal{W}_K$*-algebra minimal model.*

Crudely, a $\mathcal{W}_K$-algebra minimal model is some package of representations of the $\mathcal{W}$-algebra. In this paper, we say something about every representation in the $(K, K+N)$ $\mathcal{W}_K$-algebra minimal model.

Theorem 2.1.2 (Informal statement of Theorem 2.5.1)**.** *Fix* K *and* N *coprime. Given any representation in the* $(K, K+N)$ $\mathcal{W}_K$*-algebra minimal model, its effective* *central charge c_{eff} is equal to a value μ computed from a solution of Hitchin's equations fixed by the S^1-action in Definition 2.1.1.*

The map between representations and S^1-fixed points is through the combinatorial classifying data sets of a "cyclic K-partition of N" (see Definition 2.2.2), as shown in Figure 2.1. In §2.2, we show that there is a solution of Hitchin's equations (A, Φ) fixed by the S^1-action for every cyclic K-partition of N. In §2.3, we associate a number μ to the S^1-fixed point (A, Φ). The value μ has three interpretations (§2.3). The bulk of the new results required for the proof of the main theorem are in §2.2-2.3. In §2.4, we review some well-known facts about $\mathcal{W}_K$-algebras. The statement and proof of the main theorem, Theorem 1.5.1, make up §2.5.

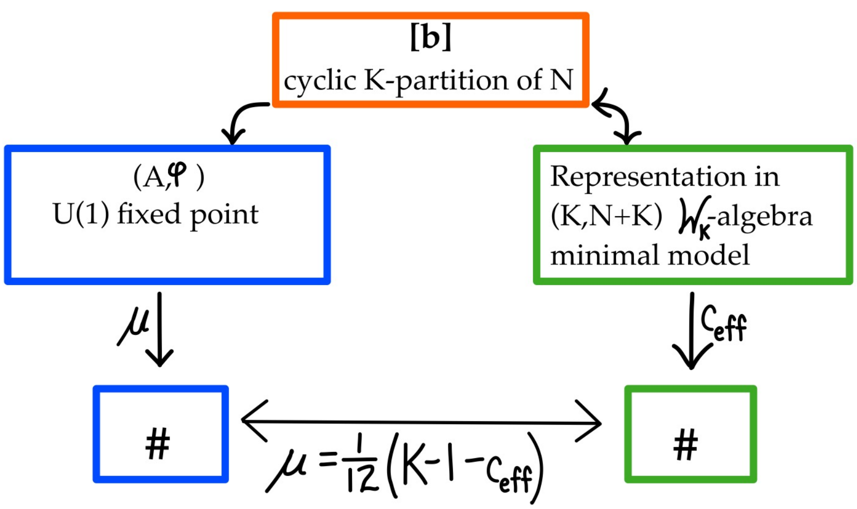

Figure 2.1: *The main theorem (Theorem 2.5.1) is a dictionary between μ and c_{eff}.*

2.2 Facts about S^1-fixed points

In this section, we prove some basic facts about solutions of Hitchin's equations fixed by the S^1-action. The S^1-action depends on K and N, and so the S^1-fixed points also depend on K and N.

Definition 2.2.1. Fix K and N coprime. A solution of Hitchin's equations (A, Φ) is **fixed by the corresponding S^1-action** if the S^1-action can be undone by a $SU(E)$-gauge transformation. In particular, for every fixed choice of $\theta \in \mathbb{R}$, there is a section $g_\theta \in \Gamma(\mathbb{CP}^1, SU(E))$ such that

$$\begin{aligned} \rho^* A &= g_\theta A g_\theta^{-1} \\ \mathrm{e}^{i\theta} \rho^* \Phi &= g_\theta \Phi g_\theta^{-1}. \end{aligned} \tag{2.4}$$

2.2.1 General facts

The integer K is the rank of the complex vector bundle E. The following lemma gives an interpretation of the integer N:

Lemma 2.2.1. *If (A, Φ) is a S^1-fixed point, then the characteristic polynomial of the Higgs field is*

$$\mathrm{char}_\Phi(x) = x^K - cz^N \mathrm{d}z^K \tag{2.5}$$

for some constant $c \in \mathbb{C}$.

Proof. We will prove that if K and N are coprime, then $\mathrm{Tr}\ \Phi^j = 0$ for $j \neq K$ and $\mathrm{Tr}\Phi^K = cz^N \mathrm{d}z$ for some constant c. Observe

$$\mathrm{Tr}\ \Phi^j = \mathrm{Tr}\left(g_\theta \Phi g_\theta^{-1}\right)^j \tag{2.6}$$

$$= \mathrm{e}^{ij\theta}\rho^* \mathrm{Tr}\, \Phi^j.$$

Consequently, $\mathrm{Tr}\Phi^j$ is a constant multiple of $z^{r_j}\mathrm{d}z^j$ for some r_j. Eq. 2.6 imposes conditions on the powers, r_j:

$$\begin{aligned} cz^{r_j}\mathrm{d}z^j &= \mathrm{Tr}\, \Phi^j \\ &= \mathrm{e}^{ij\theta}\rho^* \mathrm{Tr}\, \Phi^j \\ &= c\mathrm{e}^{ij\theta}\left(\mathrm{e}^{-i\frac{K}{K+N}\theta} z\right)^{r_j}\left(\mathrm{d}(\mathrm{e}^{-i\frac{K}{K+N}\theta} z)\right)^j \\ &= \mathrm{e}^{i\theta(j-\frac{K}{K+N}(r_j+j))}\left(cz^{r_j}\mathrm{d}z^j\right). \end{aligned} \tag{2.7}$$

Consequently, $r_j = \frac{jN}{K}$. In particular, $\mathrm{Tr}\ \Phi^K = cz^N \mathrm{d}z^K$, for some constant c. For $j \neq K$, since K and N are coprime, $r_j = \frac{jN}{K}$ is not an integer. Consequently, to be well-defined, $\mathrm{Tr}\ \Phi^j$ must vanish.

Suppose that $\mathrm{char}_\Phi(x) = x^K + a_1 x^{K-1} + \cdots + a_{K-1}x + a_K$. Consider $\mathrm{Tr}\left(\Phi^j \mathrm{char}_\Phi \Phi\right)$ for $j = 0, \cdots, K-1$. Because $\mathrm{char}_\Phi(\Phi) = 0$,

$$\begin{aligned} 0 &= \mathrm{Tr}\left(\Phi^j \mathrm{char}_\Phi(\Phi)\right) \\ &= \mathrm{Tr}\left(\Phi^{K+j} + a_1\Phi^{K-1+j} + \cdots + a_{K-1}\Phi^{1+j} + a_K\Phi^j\right) \\ &= \begin{cases} \mathrm{Tr}\left(\Phi^K + a_K \mathrm{Id}\right) & \text{if } j = 0 \\ \mathrm{Tr}\left(a_j \Phi^K\right) & \text{if } j \neq 0 \end{cases} \\ &= \begin{cases} cz^N \mathrm{d}z^K + K a_K & \text{if } j = 0 \\ a_j c^N \mathrm{d}z^K & \text{if } j \neq 0 \end{cases} \end{aligned} \tag{2.8}$$

Consequently, $a_j = 0$ for $j = 1, \cdots, K-1$, and $a_K = -\frac{c}{K} z^N \mathrm{d}z^K$. Hence, $\mathrm{char}_\Phi x = x^K - c' z^N \mathrm{d}z^K$, as claimed. □

Remark 2.2.1. ▷ We will fix the constant $c \in \mathbb{C}$ in Lemma 2.2.1 as data. ◁

Remark 2.2.2. ▷ In the case of the usual S^1-action (Eq. 2.3) on the $SU(K)$-Hitchin moduli space, any S^1-fixed point (A, Φ) of the usual S^1-action on the lies in the nilpotent cone, i.e. $\mathrm{char}_\Phi(x) = x^K$. The above lemma gives an analog of the nilpotent cone in this situation.

Smooth points of the Hitchin moduli space represent equivalence classes of irreducible solutions of Hitchin's equations. The following Corollary of Lemma 2.2.1 states that S^1-fixed points are irreducible.

Corollary 2.2.2. *If (A, Φ) is a S^1-fixed point and the constant c in Lemma 2.2.1 is non-zero, then (A, Φ) is irreducible, i.e. there is no proper subbundle of E preserved by A and Φ.*

Proof. Suppose, to the contrary, that (A, Φ) is irreducible. Then there is some proper subbundle F that is preserved by A and Φ. In particular, as block diagonal matrix in $F \oplus F^\perp$, $\Phi \in \Omega^1(C, \mathrm{End}\, E)$ has the following shape:

$$\Phi = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}. \tag{2.9}$$

Then $\mathrm{char}_\Phi x = \mathrm{char}_\alpha x \cdot \mathrm{char}_\beta x$. However, $x^K - cz^N$ is irreducible when K and N are coprime. Consequently, either $\deg \alpha = 0$ (i.e. $F = 0$) or $\deg \beta = 0$ (i.e. $F = E$). □

Lemma 2.2.3. *If (A, Φ) is a S^1-fixed point and the constant c in Lemma 2.2.1 is non-zero, then there is some $\Xi \in \Gamma(\mathrm{End}\, E)$ such that for all $\theta \in \mathbb{R}$, $g_\theta = \mathrm{e}^{\theta \Xi}$ satisfies Eq. 2.4 undoes the S^1-action.*

Proof. If (A, Φ) is irreducible, then g_θ is uniquely defined up to multiplication by a constant multiple of the identity. As a result, since rotation by $\mathrm{e}^{i\theta_1}$ followed by rotation by $e^{i\theta_2}$ is the same as rotation by $e^{i(\theta_1+\theta_2)}$, the corresponding gauge transformations also satisfy

$$g_{\theta_2} g_{\theta_1} = c g_{\theta_1+\theta_2} \tag{2.10}$$

for some constant c, which is a K^{th}-root of unity. Since c is discrete, we can choose a family $\{g_\theta\}_{\theta\in\mathbb{R}}$ such that $g_0 = \mathrm{Id}$ and

$$g_{\theta_2} g_{\theta_1} = g_{\theta_1+\theta_2}. \tag{2.11}$$

In particular $g_\theta = e^{i\theta\Xi}$ for some fixed $\Xi \in \Gamma(\mathrm{End}\, E)$. □

2.2.2 S^1-fixed points

In this section we prove that we can associate an S^1-fixed point to certain classification data ("a K-cyclic partition of N"). The precise result is stated in Theorem 2.2.4.

Definition 2.2.2.

- An **ordered K-partition of** N is an ordered K-tuple $\mathbf{b} = (b_1, \cdots, b_K)$ with $b_i \geq 0$ such that $b_1 + \cdots + b_K = N$.
- A **cyclic K-partition of** N is an equivalence class $[\mathbf{b}]$ of ordered K-partitions of N. We say two ordered K-partitions of N, $\mathbf{b}$ and $\mathbf{b}'$, are equivalent if there is a integer j such that $(b_1, \ldots, b_K) = (b'_{(1+j)(\mathrm{mod}\ K)}, \ldots, b'_{(K+j)(\mathrm{mod}\ K)})$ as ordered K-partitions of N.

Theorem 2.2.4. *Fix K and N coprime. Given a K-partition of N, $\mathbf{b} = (b_1, \cdots, b_K)$, the pair (A, Φ), given below, is a smooth solution of Hitchin's equations fixed by the*

S^1 action:

$$
\begin{aligned}
A &= \left(\frac{K+N}{2K}\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_K\end{pmatrix} + \frac{|z|}{4}\begin{pmatrix}\partial_{|z|}u_1 & & \\ & \ddots & \\ & & \partial_{|z|}u_K\end{pmatrix}\right)\left(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\bar{z}}{\bar{z}}\right) \\
\Phi &= \begin{pmatrix} & z^{b_1}|z|^{\frac{N}{K}-b_1}e^{\frac{1}{2}(u_1-u_2)} & & \\ & & \cdots & \\ & & & z^{b_{K-1}}|z|^{\frac{N}{K}-b_{K-1}}e^{\frac{1}{2}(u_{K-1}-u_K)} \\ z^{b_K}|z|^{\frac{N}{K}-b_K}e^{\frac{1}{2}(u_K-u_1)} & & & \end{pmatrix}\mathrm{d}z.
\end{aligned}
\tag{2.12}
$$

Here, the constants c_i are related to b_i by

$$b_i - \frac{N}{K} = \frac{K+N}{K}\left(c_{i+1} - c_i\right), \tag{2.13}$$

and the $\mathbb{R}$-valued functions $u_i(z) = u_i(|z|)$ solve

$$\frac{1}{4}\left(\frac{d^2}{d|z|^2} + \frac{1}{|z|}\frac{d}{d|z|}\right)u_i = |z|^{\frac{2N}{K}}\left(e^{u_i-u_{i+1}} - e^{u_{i-1}-u_i}\right). \tag{2.14}$$

As $|z| \to \infty$, the functions $u_i(|z|)$ decay to zero. The functions $|z|^{\frac{2(K+N)c_i}{K}c}\mathrm{e}^{u_i}$ are smooth at $|z| = 0$.

The S^1-action by $\mathrm{e}^{i\theta}$ is undone by

$$g_\theta = \mathrm{e}^{i\theta\mathrm{diag}(\mathbf{c})}, \tag{2.15}$$

for $\mathbf{c} = (c_1, \cdots, c_K)$ as in Eq. 2.13. If $[\mathbf{b}_1] = [\mathbf{b}_2]$ as cyclic K-partitions of N then $(A_1, \Phi_1) = g \cdot (A_2, \Phi_2)$ for gauge transformation g given by the permutation matrix corresponding to some power of $\sigma = (1\ 2\ \cdots K)$.

We note the following:

$$\lim_{|z|\to 0}|z|\frac{du_i}{d|z|} = -\frac{2(K+N)}{K}c_i \tag{2.16}$$

$$\lim_{|z|\to\infty} |z|\frac{du_i}{d|z|} = 0 \tag{2.17}$$

$$\lim_{|z|\to 0} \sum_{i=1}^{K} |z|^{\frac{2(K+N)}{K}} (\mathrm{e}^{u_i - u_{i+1}} - 1) = 0 \tag{2.18}$$

Remark 2.2.3. The somewhat strange looking results in Eq.2.16-2.18 are exactly what we'll need in §2.3.

Notation. Let $\mathbf{c} = (c_1, \cdots, c_K)$ and $\mathbf{b} = (b_1, \cdots, b_K)$. We will compactly encode Eq. 2.13 by

$$\mathbf{c} = \frac{K}{K+N} B\mathbf{b}. \tag{2.19}$$

where the entries of B are

$$B_{ij} = \frac{1}{2K}\left(-(K-1) + 2\left(\ (j-i)(\mathrm{mod}\ K)\ \right)\right). \tag{2.20}$$

The proof of Theorem 2.2.4 follows in §2.2.2. A straightforward computation shows that any such (A, Φ) is a solution of Hitchin's equations fixed by the S^1 action. The difficult thing to show is that the functions $u_i(|z|)$ exist. There are situations in which the existence of such solutions follows directly from ODE literature, as discussed in Remark 2.2.5 and 2.2.6. However, in general, we will prove the existence of such solutions using a non-abelian Hodge correspondence. Mochizuki proved a non-abelian Hodge correspondence between "good filtered Higgs bundles" and "wild harmonic bundles" [Moc10]. This is the relevant setting for us. After defining these objects, we will associate a good filtered Higgs bundle to $\mathbf{b}$, an ordered K-partition of N, which is fixed by a $\mathbb{C}^\times$-action. Mochizuki's non-abelian Hodge correspondence gives the existence and uniqueness of a harmonic metric on the filtered Higgs bundle that is "adapted" to the filtration. Because this metric is unique, it inherits certain symmetries of the underlying filtered Higgs bundle. In a unitary gauge, the triple $(\bar{\partial}_E, \varphi, h)$ corresponds to the pair (A, Φ). This pair is fixed by the S^1-action.

Remark 2.2.4. ▷ It is worth mentioning Mochizuki does something very similar in his paper "Harmonic bundles and Toda lattices with opposite sign" [Moc13]. His approach in [Moc13] motivates our approach, and we use his non-abelian Hodge correspondence in [Moc10]. In [Moc13], Mochizuki considers $\mathbb{C}^\times$-actions on filtered Higgs bundles on $\mathbb{CP}^1$ with poles at ∞ and 0. Our situation is slightly different because we do not allow any pole—even a simple pole—at zero. If we transform our S^1-fixed points into his situation, the smoothness at zero is no longer clear. Moreover, all (K, N) S^1-fixed points are identified in his picture.◁

Remark 2.2.5. ▷ For the case $K = 2$, the existence of the functions $u_i(|z|)$ follows from existing work of McCoy-Tracy-Wu [MTW77]. We know much more about the solution than we do in general. For $K = 2$, Eq. 2.14 reduces to the sinh-Gordon equation for the single function $u = u_1 = -u_2$

$$\left(\frac{d^2}{ds} + \frac{1}{s}\frac{d}{ds}\right) u = \frac{1}{2}\sinh(2u) \tag{2.21}$$

in the change of variables $s = \frac{8}{N+2}|z|^{\frac{N+2}{2}}$. The sinh-Gordon equation, in turn, transforms to a Painlevé III type equation. McCoy-Tracy-Wu extensively studied solutions of Painlevé III [MTW77]. (In their notation, we consider $\nu = 0$, $\sigma = \frac{N-2b}{N+2}$.) They give asymptotic series expansions for u and e^{-u} at 0, and decay estimates at ∞. The first term of their decay gives that

$$u(s) \to \frac{2}{\pi}\sin\left(\frac{N\pi}{2(N+2)}\right) K_0(s) \quad s \to \infty, \tag{2.22}$$

where $K_0(s)$ the modified Bessel function of the first kind. Consequently, we see that $u(|z|)$ decays like $\exp(-\frac{8}{N+2}|z|^{\frac{N+2}{2}})$ as $|z| \to \infty$. $\triangleleft$

Remark 2.2.6. $\triangleright$ In the case $K = 3$, $\mathbf{b} = (N, 0, 0)$, the coupled ODE for (u_1, u_2, u_3) also reduces to an ODE in a single equation. Take $u_1 = u$, $u_2 = 0$, $u_3 = -u$. Then get

$$\frac{1}{4}\left(\frac{d^2}{d|z|^2} + \frac{1}{|z|}\frac{d}{d|z|}\right) u = |z|^{\frac{2N}{3}}\left(e^u - e^{-2u}\right) \tag{2.23}$$

With the change of variables $s = \frac{12}{N+3}|z|^{\frac{N+3}{3}}$, this becomes the radial ODE

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right) u = e^u - e^{-2u} \tag{2.24}$$

corresponding to the Tzitzeica PDE. This ODE is known to correspond to Painlevé III. For example, taking a new change of variables $\tilde{s} = \left(\frac{5}{2}\right)^{3/2}|z|^{8/5}$, the function $\eta(\tilde{s}) = \mathrm{e}^{-u(\tilde{s})}\tilde{s}^{1/3}$ solves

$$\eta'' = \frac{1}{\eta}(\eta')^2 - \frac{1}{\tilde{s}}\eta' + \frac{1}{\tilde{s}}\eta^2 - \frac{1}{\eta}. \tag{2.25}$$

This, again, falls under the cases considered by McCoy-Tracy-Wu[MTW77]. $\triangleleft$

2.2.2.1 A good filtered Higgs bundle

In §2.1, we mentioned that the S^1-fixed points we are interested in are conjecturally elements of some twisted wild Hitchin moduli space, generalizing the wild Hitchin moduli spaces studied by Biquard-Boalch. We further noted that Mochizuki had proved a non-abelian Hodge correspondence between "good filtered Higgs bundles" and "wild harmonic bundles." In this section, we define good filtered Higgs bundles, and prove that we can associate a good filtered Higgs bundle to a cyclic K-partition of N in Proposition 2.2.5.

In [BB02], Biquard-Boalch introduce wild Hitchin moduli spaces. The Higgs bundles in their non-abelian Hodge correspondence may have high order poles. However, they impose the condition that near any such singular point, the polar part of the Higgs field is diagonalizable. In particular, there is a local holomorphic coordinate z near $p \in C$, and a local holomorphic trivialization of $(E, \bar{\partial}_E)$ in which

$$\varphi = P_n \frac{dz}{z^n} + \cdots + P_2 \frac{dz}{z^2} + P_1 \frac{dz}{z} + \text{holomorphic terms} \tag{2.26}$$

where all $P_i \in \mathfrak{su}(K)$ are diagonal. Mochizuki's "good filtered Higgs bundles" ([Moc13] §2.1.1) generalize this. The polar part of the Higgs field need not by diagonalizable. Rather, allow Higgs fields whose polar part diagonalizes on some local ramified cover $w^m = z$.

The following series of definitions culminates with the definition of a good filtered Higgs bundle. Let C be a compact complex curve and let D be a divisor locus.

Definition 2.2.3. A **meromorphic Higgs bundle** on (C, D) consists of the data $(\mathcal{E}, \varphi)$ consisting of a holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$ over C and a Higgs field φ which is $\bar{\partial}_E$-meromorphic on C and $\bar{\partial}_E$-holomorphic on $C - D$.

As soon as we allow our Higgs field to have poles along some divisor D, we must add additional data on the underlying holomorphic vector bundle over D. We take the perspective that a holomorphic vector bundle $\mathcal{E}$ is determined by its sheaf of holomorphic sections $\mathcal{O}(\mathcal{E})$. Then, the required additional data is a filtration.

Definition 2.2.4.

- A **filtered bundle** on (C, D) is a locally free $\mathcal{O}_C(D)$-module of finite rank with an increasing filtration by locally free $\mathcal{O}_C$-submodules $\{\mathcal{P}_\alpha \mathcal{E}\}_{\alpha \in \mathbb{R}}$ such that
  - $\mathcal{P}_\alpha(\mathcal{E})\Big|_{C-D} = \mathcal{O}(\mathcal{E})\Big|_{C-D}$
  - Let z be a local holomorphic coordinate on U centered at $p \in D$. Then, $\mathcal{P}_{\alpha-1}\mathcal{E}\Big|_U = z\mathcal{P}_\alpha \mathcal{E}|_U$. (Consequently, the filtration is really determined by $\alpha \in [0, 1)$.)

- A **filtered Higgs bundle** is a pair $(\mathcal{P}_\bullet \mathcal{E}, \varphi)$ consisting of a filtered bundle $\mathcal{P}_\bullet \mathcal{E}$ and a Higgs field

$$\varphi : \bigcup_{\alpha \in \mathbb{R}} \mathcal{P}_\alpha \mathcal{E} \to \left(\bigcup_{\alpha \in \mathbb{R}} \mathcal{P}_\alpha \mathcal{E} \right) \otimes K_C. \tag{2.27}$$

  (Note there are no additional compatibility conditions between the Higgs field and the filtration.)

- A filtered Higgs bundle is called **regular** if $\varphi(\mathcal{P}_\alpha \mathcal{E}) \subset \mathcal{P}_{\alpha+1}\mathcal{E} \otimes K_C$. In this case, the Higgs field is called **logarithmic**. (Note that this amounts to the Higgs field having at most simple poles on D.)

- A filtered Higgs bundle is called **unramifiedly good** if near each point $p \in D$ there is
  - a local holomorphic coordinate z,
  - a local decomposition of $\mathcal{P}_\bullet \mathcal{E} = \oplus_{i=1}^r \mathcal{P}_\bullet \mathcal{E}_i$, and
  - choice of singular type $\mathfrak{a}_i \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$

  such that

- φ respects the decomposition. (Let φ_i denote the restriction to the i^{th} piece.)
- $\varphi_i - d\mathfrak{a}_i$ is logarithmic with respect to $\mathcal{P}_\bullet\mathcal{E}_i$.

(Note that this is the condition that Biquard-Boalch impose.)

- A filtered Higgs bundle $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ on (C, D) is called **good** if near each point $p \in D$ there is a
  - a local holomorphic coordinate z on $U \ni p$, and
  - a ramified covering

$$\begin{aligned} \psi : \tilde{U} \quad &\to U \quad \subset C \\ w \quad &\mapsto \quad w^m = z \end{aligned} \tag{2.28}$$

such that $\psi^*(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ is unramifiedly good on $\tilde{U}$.

In this section, we prove that we can associate a good filtered Higgs bundle to a cyclic K-partition of N fixed by certain $\mathbb{C}^\times$ action.

Definition 2.2.5. Fix K and N. Let z be the usual holomorphic coordinate on $\mathbb{C}$.

- Let $(\bar{\partial}_E, \varphi)$ be a meromorphic Higgs bundles on $(\mathbb{CP}^1, \{\infty\})$. For ζ in the $(K+N)$-fold cover of $\mathbb{C}^\times$, define an action of ζ on $(\bar{\partial}_E, \varphi)$ by

$$\begin{aligned} z &\overset{\rho}{\mapsto} \zeta^{-\frac{K}{K+N}} z \\ \varphi &\mapsto \zeta \rho^* \varphi \end{aligned} \tag{2.29}$$

$$\bar{\partial}_E \mapsto \rho^*\bar{\partial}_E.$$

Call this action the $\mathbb{C}^\times$**-action by** ζ**.**

- A meromorphic Higgs bundle $(\bar{\partial}_E, \varphi)$ is **fixed by the** $\mathbb{C}^\times$**-action** if for each ζ there if the $\mathbb{C}^\times$-action can be undone by a $SL(E)$-gauge transformation. In particular, for every fixed choice of ζ, there is a section $g_\zeta \in \Gamma(C, SL(E))$ such that

$$\begin{aligned} \rho^*\bar{\partial}_E &= g_\zeta \bar{\partial}_E g_\zeta^{-1} \\ \zeta\rho^*\varphi &= g_\zeta \varphi g_\zeta^{-1}. \end{aligned} \tag{2.30}$$

- We say a filtered Higgs bundle $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ is **fixed by the** $\mathbb{C}^\times$**-action** if g_ζ also preserves the filtration structure at ∞, viewing g_ζ as map from $\rho^*(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ to $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$.

Proposition 2.2.5. *Given a ordered K-partition of N $\mathbf{b} = (b_1, \cdots, b_K)$, take the following meromorphic Higgs bundle on $(\mathbb{CP}^1, \{\infty\})$:*

$$\mathcal{E} = \mathcal{O}^{\oplus K} \tag{2.31}$$

$$\varphi = \begin{pmatrix} 0 & z^{b_1} & & \\ & & \ddots & \\ & & & z^{b_{K-1}} \\ z^{b_K} & & & \end{pmatrix} \mathrm{d}z. \tag{2.32}$$

There is a filtered bundle $\mathcal{P}_\bullet\mathcal{E}$ such that

- $\mathcal{P}_\bullet\mathcal{E}\Big|_{\mathbb{CP}^1-\{\infty\}} = \mathcal{O}(\mathcal{E})\Big|_{\mathbb{CP}^1-\{\infty\}}$*, where $\mathcal{O}(\mathcal{E})$ denotes the sheaf of holomorphic sections of $\mathcal{E}$,*

- $(\mathcal{P}_\bullet\mathcal{E},\varphi)$ *is a good filtered Higgs bundle, in the sense of Definition 2.2.4, and*

- $(\mathcal{P}_\bullet\mathcal{E},\varphi)$ *is fixed by the* $\mathbb{C}^\times$*-action in the sense of Definition 2.2.5.*

Proof. Take

$$g_\zeta = \begin{pmatrix} \zeta^{c_1} & & \\ & \ddots & \\ & & \zeta^{c_K} \end{pmatrix}, \tag{2.33}$$

where c_i are related by b_i by

$$b_i - \frac{N}{K} = \frac{K+N}{K}\left(c_{i+1} - c_i\right), \tag{2.34}$$

as in Eq. 2.13. Note that the $\mathbb{C}^\times$-action on the meromorphic Higgs bundle $(\mathcal{E},\varphi)$ is undone by g_ζ.

We now seek to show that there is a filtered bundle $\mathcal{P}_\bullet\mathcal{E}$ such that $(\mathcal{P}_\bullet\mathcal{E},\varphi)$ is a good filtered Higgs bundle fixed by the $\mathbb{C}^\times$-action. We first show that $(\mathcal{P}_\bullet\mathcal{E},\varphi)$ is a good filtered Higgs bundle. Let $z' = \frac{1}{z}$ be the usual holomorphic coordinate at ∞. Take the unit disk $\mathbb{D}$ as a local neighborhood of ∞. Let $(\widetilde{\mathbb{D}}, w')$ be the K-fold cover of $(\mathbb{D}, z')$ with $\psi : w' \mapsto (w')^K = z'$. Consider the bundle $\psi^*\mathcal{E}$. The following global change of holomorphic trivialization, g, makes sense on $\psi^*\mathcal{E}\Big|_{C-\{0,\infty\}}$:

$$g = \begin{pmatrix} z^{\frac{K+N}{K}c_1} & & & & \\ & z^{\frac{K+N}{K}c_2} & & & \\ & & z^{\frac{K+N}{K}c_3} & & \\ & & & \ddots & \\ & & & & z^{\frac{K+N}{K}c_K} \end{pmatrix} \begin{pmatrix} \zeta_K & \zeta_K^2 & \cdots & \zeta_K^{K-1} & 1 \\ \zeta_K^2 & \zeta_K^4 & \cdots & \zeta_K^{2(K-1)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \zeta_K^{K-1} & \zeta_K^{2(K-1)} & \cdots & \zeta_K^{(K-1)^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \tag{2.35}$$

where $\zeta_K = \mathrm{e}^{2\pi i/K}$. Also note that Det g is constant, so that by rescaling g we can make a determinant one transformation. Note that

$$g^{-1}\varphi g = \begin{pmatrix} \zeta_K z^{\frac{K+N}{K}} & & & \\ & \zeta_K^2 z^{\frac{K+N}{K}} & & \\ & & \ddots & \\ & & & \zeta_K^K z^{\frac{K+N}{K}} \end{pmatrix} \mathrm{d}z. \tag{2.36}$$

Let v_i be this basis. One can see that φ maps $\langle v_i \rangle$ to $\langle v_i \rangle$. We define the filtration $\mathcal{P}_\bullet \mathcal{E}$ by a filtration on its lift $\mathcal{P}_\bullet \psi^* \mathcal{E}$. Take the trivial filtration induced by declaring that the basis elements v_i are in $\mathcal{P}_0 \psi^* \mathcal{E}$. We now claim that $(\mathcal{P}_\bullet \psi^* \mathcal{E}, \psi^* \varphi)$ on $\widetilde{\mathbb{D}}$ is an unramifiedly good filtered Higgs bundle. Rewriting Eq. 2.36 in holomorphic coordinate on $\widetilde{\mathbb{D}}$, note that

$$\psi^*\varphi = \begin{pmatrix} \zeta_K (w')^{-(K+N)} & & & \\ & \zeta_K^2 (w')^{-(K+N)} & & \\ & & \ddots & \\ & & & \zeta_K^K (w')^{-(K+N)} \end{pmatrix} \mathrm{d}z. \tag{2.37}$$

Consequently, if we take

$$\mathfrak{a}_i = \frac{\zeta_K^i}{1-(K+N)} (w')^{1-(K+N)}, \tag{2.38}$$

then

$$\psi^*\varphi - \begin{pmatrix} d\mathfrak{a}_1 & & & \\ & d\mathfrak{a}_2 & & \\ & & \ddots & \\ & & & d\mathfrak{a}_K \end{pmatrix} = 0. \tag{2.39}$$

Consequently, indeed, $(\mathcal{P}_\bullet \psi^* \mathcal{E}, \psi^* \varphi)$ is unramifiedly good filtered Higgs bundle, in the sense of Definition 2.2.4.

Not every filtration $\mathcal{P}_\bullet \psi^* \mathcal{E} \to \widetilde{\mathbb{D}}$ arises as as a pullback of a filtration on $\mathcal{P}_\bullet \mathcal{E} \to \mathbb{D}$. However, this filtration does. This is because there is an obvious ambiguity in the definition of the matrix g (Eq. 2.35) diagonalizing φ. We have the freedom to arbitrarily permute the basis vectors; consequently, the basis vectors must be treated equally in the filtration, for example, by taking all v_i in the zero-weight piece. (Note that Mochizuki proves a similar result in a slightly different similar situation in Lemma 3.9 [Moc13].) $\square$

2.2.2.2 A harmonic bundle

Given a good filtered Higgs bundle $(\mathcal{P}_\bullet\mathcal{E})$, Mochizuki proves that there is a harmonic metric h adapted to the filtration. This metric is unique up to rescaling by a constant.

Definition 2.2.6.

- A metric on a good filtered Higgs bundle on (C, D) is a hermitian metric h on the holomorphic vector bundle over $C - D$, $\mathcal{E}\Big|_{C-D}$.
- A hermitian metric h determines a filtration $\mathcal{P}^h_\bullet\mathcal{E}$ based on the grown of holomorphic sections of $\mathcal{E}\Big|_{C-D}$. Let z be a holomorphic coordinate centered at $p \in D$. A holomorphic section s is in $\mathcal{P}^h_\alpha\mathcal{E}$ if

$$\|s\|_h = O\left(|z|^{-(\alpha+\varepsilon)}\right) \text{ for all } \varepsilon > 0. \tag{2.40}$$

- We say a hermitian metric h is **adapted to the filtration** $\mathcal{P}_\bullet\mathcal{E}$ if the induced filtration from h agrees with the original filtration, i.e. $\mathcal{P}^h_\bullet\mathcal{E} = \mathcal{P}_\bullet\mathcal{E}$.
- A hermitian metric on a filtered Higgs bundle is **harmonic** if it solves Hitchin's

equations on $C - D$

$$F_{D(\bar{\partial}_E,h)} + [\varphi, \varphi^{\dagger_h}] = 0, \tag{2.41}$$

where D is the Chern connection associated to the pair $(\bar{\partial}_E, h)$ and $\varphi^{\dagger_h}$ is the hermitian adjoint of φ with respect to h.

We are interested in h which induce some fixed harmonic metric on $\mathrm{Det}\mathcal{E} \cong \mathcal{O}$. Consequently, h is actually determined uniquely in the $SL(K,\mathbb{C})$— rather than $GL(K,\mathbb{C})$ setting. In this section we investigate this unique adapted harmonic metric h.

From the uniqueness of h, see that h is diagonal: We saw in Proposition 2.2.5 that $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ is fixed by a $\mathbb{C}^\times$-action. Consequently, g_ζ is a filtered Higgs bundle isomorphism mapping $\rho^*(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ to $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$. By the the uniqueness of the adapted harmonic metric g_ζ also maps $\rho^* h$ to h via

$$\rho^* h = g_\zeta h g_\zeta^\dagger. \tag{2.42}$$

We restrict our attention to unit length ζ and take $\zeta = \mathrm{e}^{i\theta}$ for $\theta \in \mathbb{R}/(K+N)\mathbb{Z}$. For simplicity let $g_\theta = g_{\mathrm{e}^{i\theta}}$. From Proposition 2.2.5, recall that the gauge transformation g_θ which undoes the S^1-action is

$$g_\theta = \begin{pmatrix} \mathrm{e}^{i\theta c_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\theta c_K} \end{pmatrix} \tag{2.43}$$

where $\mathbf{c} = (c_1, \cdots, c_K)$ was related to $\mathbf{b} = (b_1, \cdots, b_K)$ by Eq. 2.34.

Proposition 2.2.6. *Let K and N be coprime integers. Let $\mathbf{b}$ be an ordered K-partition of N, and let $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ be the associated good filtered Higgs bundle given in Proposition 2.2.5. Then the adapted harmonic metric h is diagonal in the basis of Proposition 2.2.5. Moreover, the diagonal entries h_{ii} are real-valued functions of $|z|$ alone.*

Proof. By the uniqueness, see that

$$\rho^* h = g_\theta h g_\theta^\dagger. \tag{2.44}$$

If h_{ij} denotes the (i, j) entry of h then,

$$h_{ij}\left(e^{-i\theta\frac{K}{K+N}}z\right) = h_{ij}(z)\left(e^{i\theta(c_i-c_j)}\right). \tag{2.45}$$

Take $\theta = 2\pi\frac{K+N}{K}$ so $-i\theta\frac{K}{K+N} = -2\pi i$. Then

$$h_{ij}(z) = h_{ij}\left(e^{-2\pi i}z\right) = h_{ij}(z)\left(e^{2\pi i\frac{K+N}{K}(c_i-c_j)}\right). \tag{2.46}$$

This can only be true if $h_{ij} = 0$ or $\frac{K+N}{K}(c_i - c_j) \in \mathbb{Z}$. However, from Eq. 2.34,

$$\frac{K+N}{K}(c_j - c_i) \quad = \quad -(j-i)\frac{N}{K} + \sum_{m=i}^{j-1} b_k. \tag{2.47}$$

Because b_k are integers, if N and K are coprime, $\frac{K+N}{K}(c_i - c_j)$ is an integer if, and only if, $i = j$. Consequently, h is diagonal.

Looking at the diagonal entries, see from Eq. 2.45 that $h_{ii}(z) = h_{ii}(|z|)$. Because h is hermitian, see that the entries h_{ii} are real-valued. $\square$

We get boundary conditions at ∞ because h is adapted to the filtration and boundary conditions at 0 because h is smooth:

Proposition 2.2.7. *Let h be the unique adapted metric. From Proposition 2.2.6, h is diagonal in the gauge of Proposition 2.2.5, consequently write h as*

$$h = \begin{pmatrix} |z|^{\frac{2(K+N)c_1}{K}} e^{u_1} & & \\ & \ddots & \\ & & |z|^{\frac{2(K+N)c_K}{K}} e^{u_K} \end{pmatrix} \tag{2.48}$$

where $u_i(z) = u_i(|z|)$ and c_i are related to b_i by Eq. 2.34.

Then,

- *The functions $u_i : \mathbb{R}^{>0} \to \mathbb{R}$ solve*

$$\frac{1}{4}\left(\frac{d^2}{d|z|^2} + \frac{1}{|z|}\frac{d}{d|z|}\right) u_i = |z|^{\frac{2N}{K}} \left(e^{u_i - u_{i+1}} - e^{u_{i-1} - u_i}\right) \tag{2.49}$$

- *The function u_i decays to 0 as $|z| \to \infty$.*

- *Near 0, $u_i \sim -\frac{2(K+N)c_i}{K} \log|z|$.*

We additionally note the following:

$$\lim_{|z|\to 0} |z|\frac{du_i}{d|z|} = -\frac{2(K+N)}{K}c_i \tag{2.50}$$

$$\lim_{|z|\to \infty} |z|\frac{du_i}{d|z|} = 0 \tag{2.51}$$

$$\lim_{|z|\to 0} |z|^{\frac{2(K+N)}{K}} (\mathrm{e}^{u_i - u_{i+1}} - 1) = 0 \tag{2.52}$$

Remark 2.2.7. ▷ Note that under the change of variables $\rho^2 = \frac{2K}{N+K}|z|^{\frac{2(N+K)}{K}}$ we get the coupled system of ODE which is the radial version of the coupled system of PDE known as "2D cyclic affine Toda lattice with opposite sign"

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right) u_i = e^{u_i - u_{i+1}} - e^{u_{i-1} - u_i}. \tag{2.53}$$

◁

Proof. To see that u_i solve Eq. 2.49, we simply compute. In the basis of Proposition 2.2.5, the curvature $F_{D(\bar{\partial}_E,h)}$ is a diagonal matrix with (i,i)-entry given by

$$\left(F_{D(\bar{\partial}_E,\varphi)}\right)_{ii} = -\frac{1}{4|z|}\frac{d}{d|z|}\left(|z|\frac{du_i}{d|z|}\right)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.54}$$

Similarly, the quantity $\varphi \wedge \varphi^{\dagger_h}$ is a diagonal matrix with (i,i)-entry given by

$$\left(\varphi \wedge \varphi^{\dagger}\right)_{ii} = |z|^{\frac{2N}{K}} e^{u_i - u_{i+1}} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.55}$$

It follows that u_i solve Eq. 2.49.

The properties of u_i near zero follows the smoothness of the harmonic metric. To see Eq. 2.50, i.e.

$$\lim_{|z|\to 0} |z|\frac{d}{d|z|}u_i = -\frac{2(K+N)c_i}{K}, \tag{2.56}$$

note that because h is smooth at $0 \in \mathbb{C}$, $\frac{d}{d|z|}\Big|_{|z|=0} |z|^{\frac{2(K+N)c_i}{K}} e^{u_i} = 0$. Hence,

$$\begin{aligned} 0 &= \frac{d}{d|z|}\Big|_{|z|=0} |z|^{\frac{2(K+N)c_i}{K}} e^{u_i} \\ &= |z|^{\frac{2(K+N)c_i}{K}} e^{u_i}\left(\frac{2(K+N)c_i}{K}\frac{1}{|z|} + \frac{d}{d|z|}u_i\right), \end{aligned} \tag{2.57}$$

and Eq. 2.50/2.56 follows. To see Eq. 2.52, i.e.

$$\lim_{|z|\to 0} |z|^{\frac{2(K+N)}{K}} (\mathrm{e}^{u_i - u_{i+1}} - 1) = 0, \tag{2.58}$$

note that $\mathrm{e}^{u_i} = f_i |z|^{\frac{2(K+N)c_i}{K}}$ where $f_i = f_i(|z|)$ is a smooth function that does not vanish at 0. Consequently,

$$|z|^{\frac{2(K+N)}{K}} (\mathrm{e}^{u_i - u_{i+1}}) = |z|^{\frac{2(K+N)}{K}} \left(|z|^{\frac{2(K+N)(c_{i+1}-c_i)}{K}} \frac{f_i}{f_{i+1}}\right) \tag{2.59}$$

$$= \quad |z|^{\frac{2(K+N)}{K}} \left(|z|^{2b_i - \frac{2N}{K}} \frac{f_i}{f_{i+1}} \right)$$
$$= \quad |z|^{2(1+b_i)} \frac{f_i}{f_{i+1}}.$$

Because f_{i+1} is smooth and does not vanish at $|z| = 0$ and b_i is non-negative, then the expression in Eq. 2.59 vanishes at $|z| = 0$.

To see the boundary conditions at ∞, we pass to the ramified K-fold cover of the disk $\mathbb{D}$ around ∞ in which $\psi^*\varphi$ is diagonal, as written in Eq. 2.36. Let g be the gauge transformation, given in Eq. 2.35, which diagonalizes φ near ∞. Then

$$g^{-1}h\left(g^{-1}\right)^{\dagger} = \begin{pmatrix} \zeta_K & \zeta_K^2 & \cdots & 1 \\ \zeta_K^2 & \zeta_K^4 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{u_1} & & & \\ & e^{u_2} & & \\ & & \ddots & \\ & & & e^{u_K} \end{pmatrix} \begin{pmatrix} \zeta_K & \zeta_K^2 & \cdots & 1 \\ \zeta_K^2 & \zeta_K^4 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}^{-1\,\dagger}. \tag{2.60}$$

In this basis, ψ^*h is adapted to the trivial filtration. Consequently, ψ^*h is bounded, hence e^{u_i} is bounded above and below by some constant. Since there is no logarithmic term in the expansion of u_i at ∞,

$$\lim_{z\to\infty} |z|\frac{d}{d|z|}u_i = 0, \tag{2.61}$$

hence Eq. 2.51 holds. □

Remark 2.2.8. ▹ The powers of $|z|$ appearing in the the harmonic map h at ∞ can also be explained via the semi-flat metric. The semi-flat metric will appear again in §2.3.1.

Definition 2.2.7. The **semi-flat metric** h_{sf} is the (singular) metric such that $F_{D(\bar{\partial}_E, h_{\mathrm{sf}})} = 0$ and $[\varphi, \varphi^{\dagger_{h_{\mathrm{sf}}}}] = 0$. Here, $D(\bar{\partial}_E, h_{\mathrm{sf}})$ is the Chern connection associated to the pair and $\varphi^{\dagger_{h_{\mathrm{sf}}}}$ is the hermitian adjoint of φ with respect to h_{sf}.

Given an ordered K-partition of N, let $(\bar{\partial}_E, \varphi)$ be the associated meromorphic Higgs bundle in Proposition 2.2.5. The semi-flat metric is

$$h_{\mathrm{sf}} = \begin{pmatrix} |z|^{\frac{2(K+N)c_1}{K}} & & \\ & \ddots & \\ & & |z|^{\frac{2(K+N)c_K}{K}} \end{pmatrix}. \tag{2.62}$$

Note that these are the same powers of $|z|$ appearing in Eq. 2.48. $\triangleleft$

2.2.2.3 Proof of Theorem 2.2.4

Proof of Theorem 2.2.4. The pair (A, Φ) in the statement of Theorem 2.2.4 is related to the triple $(\bar{\partial}_E, \varphi, h)$ in Propositions 2.2.5, 2.2.6, 2.2.7 by $A^{0,1} = h^{1/2} \circ \bar{\partial}_E \circ h^{-1/2}$ and $\Phi = h^{1/2}\varphi h^{-1/2}$. The pair (A, Φ) is a solution of Hitchin's equations because $(\bar{\partial}_E, \varphi, h)$ is a solution of Hitchin's equations. The pair (A, Φ) is an S^1-fixed point because the triple $(\bar{\partial}_E, \varphi, h)$ is fixed by $S^1 \subset \mathbb{C}^\times$. Take g_θ, as in Proposition 2.2.5. Note that in the basis of Proposition 2.2.5, h and g_θ are both diagonal, and consequently commute. To show that (A, Φ) is an S^1-fixed point we compute:

$$\begin{aligned} \rho^* A^{0,1} &= \rho^* h^{1/2} \circ \rho^* \bar{\partial}_E \circ \rho^* h^{-1/2} \\ &= h^{1/2} \circ g_\theta \bar{\partial}_E g_\theta^{-1} \circ h^{-1/2} \\ &= g_\theta \left(h^{1/2} \circ \bar{\partial}_E \circ h^{-1/2} \right) g_\theta^{-1} \\ &= g_\theta A^{0,1} g_\theta^{-1}. \end{aligned} \tag{2.63}$$

$$\begin{aligned} e^{i\theta} \rho^* \Phi &= \left(\rho^* h^{1/2} \right) \left(e^{i\theta} \rho^* \varphi \right) \left(\rho^* h^{-1/2} \right) \\ &= h^{1/2} \left(g_\theta \varphi g_\theta^{-1} \right) h^{-1/2} \\ &= g_\theta \left(h^{1/2} \varphi h^{-1/2} \right) g_\theta^{-1} \end{aligned} \tag{2.64}$$

$$= \quad g_\theta \Phi g_\theta^{-1}.$$

The properties of u_i in Eq. 2.16-2.18 are the same properties listed in Eq. 2.50-2.52 of Proposition 2.2.7. □

2.3 Three numbers

In [Hit87], Hitchin associated a number μ to a solution of Hitchin's equation's (without any poles) fixed by the S^1-action

$$\begin{aligned} A &\rightarrow A \\ \Phi &\rightarrow e^{i\theta}\Phi. \end{aligned} \tag{2.65}$$

The value μ has three interpretations:

- The L^2-norm of Φ: Define $\mu(A,\Phi) = \|\Phi\|_{L^2}^2 = 2i\int_C \mathrm{Tr}\ \Phi\Phi^\dagger$.
- The value of the moment map generated by the S^1-action: The moment map for the S^1-action is $-\frac{1}{2}\|\Phi\|_{L^2}^2$. Consequently, $\mathrm{d}\mu = -2\iota_X\omega_I$.
- The shifted degree of the line bundle L: If (A,Φ) is a S^1-fixed point then there is a gauge $E = L \oplus L^*\Lambda^2 E$ in which the Higgs field has the shape $\begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$. Moreover, $\mu(A,\Phi) = \pi(\deg L - \frac{1}{2})$.

In our case—now looking again at solutions of Hitchin's equations on $\mathbb{CP}^1$ with poles at infinity— we compute the values of three functions μ_1, μ_2, and μ_3 at a solution of Hitchin's equations (A,Φ) fixed by the S^1-fixed point in Eq. 2.29. These come, respectively, as a renormalized L^2-norm, as an expression involving the formal parabolic degrees of certain line bundles, and from a formal moment map.

Throughout this section, we will compare the story between Hitchin's usual S^1-action and our S^1-action.

2.3.1 μ_1 from a norm of the Higgs field

In [Hit87], Hitchin computed the value of $2i \int_{\mathbb{C}} \operatorname{Tr} \Phi \wedge \Phi^\dagger$ at a S^1-fixed point (A, Φ). In our case, this would be infinite. However, let $(A_{\mathrm{sf}}, \Phi_{\mathrm{sf}})$ be unitary pair associated to the triple $(\bar{\partial}_E, \varphi, h_{\mathrm{sf}})$. (The semi-flat metric h_{sf} was defined in Definition 2.2.7.) Because h_{sf} is the semi-flat metric $F_{A_{\mathrm{sf}}} = 0$ and $[\Phi_{\mathrm{sf}}, \Phi_{\mathrm{sf}}^\dagger] = 0$. We compute

$$\mu_1(A, \Phi) = i \int \operatorname{Tr} \left(\Phi \wedge \Phi^\dagger - \Phi_{\mathrm{sf}} \wedge \Phi_{\mathrm{sf}}^\dagger \right). \tag{2.66}$$

This can be viewed as a regularized L^2-norm. Further note, that the second term is just

$$\Phi_{\mathrm{sf}} \wedge \Phi_{\mathrm{sf}}^\dagger = |z|^{\frac{2N}{K}} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.67}$$

Proposition 2.3.1. *Given a ordered K-partition of N $\mathbf{b}$, let (A, Φ) be associated S^1-fixed point from Theorem 2.2.4. Moreover, suppose that*

$$\lim_{|z| \to \infty} \sum_{i=1}^{K} |z|^{\frac{2(N+K)}{K}} (e^{u_i - u_{i+1}} - e^{u_{i-1} - u_i}) = 0. \tag{2.68}$$

Then

$$\mu_1(A, \Phi) = \frac{K\pi}{(K+N)} \|B\mathbf{b}\|^2. \tag{2.69}$$

Remark 2.3.1. The author expects that the additional assumption made in Eq. 2.68 is automatically satisfied. Note that Eq. 2.68 is certainly satisfied for the special low rank cases mentioned in Remarks 2.2.5 & 2.2.6.

Some consequences of the fact that h is an adapted, harmonic metric are summarized in the statement of Proposition 2.2.7. Eq. 2.68 should hold as an additional, but perhaps more delicate, consequence. To show that Eq. 2.68 is automatically satisfied, it would suffice to show that $u_i - u_{i+1}$ decays faster than $|z|^{-\frac{N+K}{K}}$ near ∞. Then, near ∞,

$$|z|^{\frac{2(N+K)}{K}} \sum_{i=1}^{K} \left(e^{u_i - u_{i+1}} - 1\right) \tag{2.70}$$

would have terms $\left(|z|^{\frac{N+K}{K}}(u_i - u_{i+1})\right)^2$ quadratic in $u_i - u_{i+1}$, in addition to other higher order terms in $u_i - u_{i+1}$.

In the change of variables $\rho^2 = \frac{2K}{N+K}|z|^{\frac{2(N+K)}{K}}$ in Remark 2.2.7, and defining $w_i = u_i - u_{i+1}$, the functions w_i solve

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dw_i}{d\rho}\right) = \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right) w_i = 2e^{w_i} - e^{w_{i+1}} - e^{w_{i-1}}. \tag{2.71}$$

The claim is that w_i should decay faster than $\frac{1}{\rho}$ near ∞. (Note that in the special low rank cases mentioned above, we actually get much better decay. In particular, w_i decays like $e^{-\rho^{1-\varepsilon}}$.) Note that w_i sum to zero, are bounded near ∞, and behave like $w_i \sim \left(b_i - \frac{N}{K}\right) \log|z|$ near 0.

Proof of Proposition 2.3.1.

$$\mu_1 \quad = \quad i \int \operatorname{Tr}\left(\Phi \wedge \Phi^\dagger - \Phi_{\mathrm{sf}} \wedge \Phi_{\mathrm{sf}}^\dagger\right) \tag{2.72}$$

$$
\begin{aligned}
&\overset{Eq.\ 2.55}{=} && i\int_0^{2\pi}\int_0^{\infty}|z|^{2N/K}\sum_{i=1}^{K}\left(e^{u_i-u_{i+1}}-1\right)\left(-2i|z|d|z|\wedge d\vartheta\right)\\
&= && 4\pi\int_0^{\infty}\sum_{i=1}^{K}\left(e^{u_i-u_{i+1}}-1\right)d\left(\frac{K}{2(K+N)}|z|^{\frac{2(K+N)}{K}}\right)\\
&= && 4\pi\left[\sum_{i=1}^{K}\left(e^{u_i-u_{i+1}}-1\right)\frac{K}{2(K+N)}|z|^{\frac{2(K+N)}{K}}\right|_0^{\infty}\\
& && -4\pi\int_0^{\infty}\sum_{i=1}^{K}d\left(e^{u_i-u_{i+1}}-1\right)\frac{K}{2(K+N)}|z|^{\frac{2(K+N)}{K}}\\
&\overset{Eq.\ 2.18,2.68}{=} && -4\pi\int_0^{\infty}\frac{K}{2(K+N)}\sum_{i=1}^{K}|z|^2\frac{du_i}{d|z|}|z|^{\frac{2N}{K}}\left(e^{u_i-u_{i+1}}-e^{u_{i-1}-u_i}\right)d|z|\\
&\overset{Eq.\ 2.14}{=} && -4\pi\int_0^{\infty}\frac{K}{2(K+N)}\sum_{i=1}^{K}|z|^2\frac{du_i}{d|z|}\frac{1}{4|z|}d\left(|z|\frac{du_i}{d|z|}\right)\\
&= && -\frac{\pi K}{2(K+N)}\int_0^{\infty}\sum_{i=1}^{K}\frac{1}{2}d\left(|z|\frac{du_i}{d|z|}\right)^2\\
&= && -\frac{\pi K}{4(K+N)}\left[\sum_{i=1}^{K}\left(|z|\frac{du_i}{d|z|}\right)^2\right|_0^{\infty}\\
&\overset{Eq.\ 2.16,2.17}{=} && \frac{\pi K}{4(N+K)}\left(\frac{2(K+N)}{K}c_i\right)^2\\
&\overset{Eq.\ 2.19}{=} && \frac{\pi K}{N+K}\left\|B\mathbf{b}\right\|^2.
\end{aligned}
$$

$\square$

2.3.2 μ_2 from parabolic degrees

In this section, we compute a number μ_2 at an S^1-fixed point which we interpret in terms of the parabolic degrees of certain line subbundles $\mathcal{L}_i$ of the holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$.

Given a holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$ with Hermitian metric h over a compact complex curve, we have the following correspondence between the algebraic degree (on the left-hand side) and the analytic degree (on the right-hand side):

$$\deg(\mathcal{E}) = \frac{i}{2\pi} \int_C \mathrm{Tr}\ F_D \tag{2.73}$$

where D is the Chern connection associated to $(\bar{\partial}_E, h)$. For a holomorphic subbundle $\mathcal{F}$, and projection map $\pi : E \to F$, the degree is

$$\deg(\mathcal{F}) = \frac{i}{2\pi} \int_C \mathrm{Tr}(\pi F_D) - |\bar{\partial}_E \pi|^2. \tag{2.74}$$

A similar correspondence is true in the case of filtered holomorphic vector bundles. Given a filtered holomorphic bundle, there is a corresponding parabolic bundle by considering only the weights $\alpha \in [0, 1)$ where the filtration $\mathcal{P}_\bullet \mathcal{E}$ jumps. Let α_j be these weights and let $F_j \mathcal{E} = \mathcal{P}_{\alpha_j} \mathcal{E}$.

Definition 2.3.1. Let $\mathcal{E} \to C$ be a holomorphic vector bundle over a compact complex curve, with marked points $D = \{p_1, \cdots, p_n\}$. A **parabolic structure** on E consists of the following data at each marked point:

- a filtration $F^i_\bullet \mathcal{E}$ at each p_i with $\mathcal{E}_{p_i} = F^i_1 \mathcal{E} \supsetneq \cdots F^i_{m_i} \mathcal{E} \supsetneq 0$
- a system of weights $\alpha^i_\bullet$ with $0 \le \alpha^i_1 < \cdots < \alpha^i_{m_i} < 1$

The (algebraic) **parabolic degree** of $\mathcal{E}$ is

$$\mathrm{pdeg}\mathcal{E} = \deg\mathcal{E} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha^i_j \left(\dim F^i_j \mathcal{E} - \dim F^i_{j+1} \mathcal{E}\right) \tag{2.75}$$

If h is a hermitian metric adapted to the parabolic structure then, we have a similar formula relating the algebraic parabolic degree and analytic parabolic degree. For a holomorphic bundle $\mathcal{E}$ with parabolic structure,

$$\mathrm{pdeg}(\mathcal{E}) = \frac{i}{2\pi}\int_C \mathrm{Tr}\ F_D \tag{2.76}$$

where D is the Chern connection associated to $(\bar{\partial}_E, h)$. For a holomorphic subbundle $\mathcal{F}$,

$$\deg(\mathcal{F}) = \frac{i}{2\pi}\int_C \mathrm{Tr}(\pi F_D) - |\bar{\partial}_E \pi|^2. \tag{2.77}$$

(See [Sim90] p.750.)

Lemma 2.3.2. *Given* $\mathbf{b}$*, an ordered* K*-partition of* N*, let* (A,Φ) *be a* S^1*-fixed point given in Theorem 2.2.4. Let* $\{e_i\}$ *be the basis in Eq. 2.12. Let* $\mathcal{L}_i$ *be the holomorphic line bundle spanned by* $\langle e_i \rangle$*. (Note that* $\mathcal{L}_i$ *is holomorphic because* $\bar{\partial}_A = A^{0,1}$ *does not map* out of $\mathcal{L}_i$*.) Then, the analytic parabolic degree of* $\mathcal{L}_i$ *is*

$$\mathrm{pdeg}(\mathcal{L}_i) = -\frac{K+N}{K}c_i = -\left(B\mathbf{b}\right)_i. \tag{2.78}$$

Proof. In the basis $\{e_j\}$ in Eq. 2.12, π is the matrix with a single nonzero entry 1 in the (j,j) spot. Consequently, $\bar{\partial}_A \pi = 0$. Then,

$$\begin{aligned}
\mathrm{pdeg}(\mathcal{L}_i) &= \frac{i}{2\pi}\int_{\mathbb{CP}^1} \pi F_A \\
&\overset{Eq.\ 2.54}{=} \frac{i}{2\pi}\int_{\mathbb{CP}^1} -\frac{1}{4|z|}\frac{d}{d|z|}\left(|z|\frac{du_i}{d|z|}\right)\mathrm{d}z\wedge \mathrm{d}\bar{z} \\
&= \frac{i}{2\pi}\int_{\mathbb{CP}^1} -\frac{1}{4|z|}\frac{d}{d|z|}\left(|z|\frac{du_i}{d|z|}\right)(-2i|z|\mathrm{d}|z|\mathrm{d}\vartheta) \\
&= \frac{-\pi}{2\pi}\int_0^\infty \frac{d}{d|z|}\left(|z|\frac{du_i}{d|z|}\right)\mathrm{d}|z|
\end{aligned}$$

$$\overset{Eq.\ 2.16,\ 2.17}{=} \quad -\frac{K+N}{K} c_i$$

□

Corollary 2.3.3. *Given* $\mathbf{b}$*, an ordered* K*-partition of* N*, let* (A, Φ) *be the associated* S^1*-fixed point from Theorem 2.2.4. Define* $\mu_2(A, \Phi)$ *to be*

$$\mu_2(A, \Phi) := \sum_{i=1}^{K} \left(\mathrm{pdeg}(\mathcal{L}_i)\right)^2. \tag{2.79}$$

Then,

$$\mu_2(A, \Phi) = \|B\mathbf{b}\|^2. \tag{2.80}$$

2.3.3 μ_3 from a moment map

In this section, we compute the value of

$$\begin{aligned}\mu_3(A, \Phi) \quad = \quad & \frac{1}{4(N+K)} \int_{\mathbb{CP}^1} \mathrm{Tr}\left(-2N\Phi \wedge \Phi^\dagger + iK\left(\Phi \wedge \partial_\vartheta \Phi^\dagger - \partial_\vartheta \Phi \wedge \Phi^\dagger\right)\right. \\ & \left. -2K\left(-A^{1,0} \wedge A^{0,1}\right) - iK\left(-A^{1,0} \wedge \partial_\vartheta A^{0,1} + \partial_\vartheta A^{1,0} \wedge A^{0,1}\right)\right)\end{aligned} \tag{2.81}$$

at S^1-fixed points in Theorem 2.2.4. We explain the shape of μ_3 by relating it to the formal moment map generated by the S^1-action in Definition 2.1.1. Note that we make no claims that any of the following integrals make sense— except for the value μ_3 at a S^1-fixed point appearing in Theorem 2.2.4.

Let $\mathcal{N} = \mathcal{A} \times \Omega^{1,0}(\mathbb{CP}^1, \mathrm{End}\ E)$ be the affine space of all pairs (A, Φ) of A, a unitary connection on E and Higgs fields $\Phi \in \Omega^{1,0}(\mathbb{CP}^1, \mathrm{End}\ E)$. For now, we place no additional conditions on the pair (A, Φ).

Given a pair $(A, \Phi) \in \mathcal{N}$, the tangent space can be identified with

$$T_{(A,\Phi)}\mathcal{N} \cong \Omega^{0,1}(\mathbb{CP}^1, \mathrm{End}\ E) \oplus \Omega^{1,0}(\mathbb{CP}^1, \mathrm{End}\ E \otimes \mathbb{C}) \tag{2.82}$$

since $(\mathcal{A} - A)^{0,1} = \Omega^{0,1}(\mathbb{CP}^1, \mathrm{End}\ E \otimes \mathbb{C})$. On $T_{(A,\Phi)}\mathcal{N}$, formally define the following metric :

$$\begin{aligned} g\left((\dot{A}_1{}^{0,1}, \dot{\Phi}_1), (\dot{A}_2{}^{0,1}, \dot{\Phi}_2)\right) &= \frac{1}{2}\int_{\mathbb{CP}^1} \mathrm{Tr}\left((\dot{A}_1{}^{0,1})^\dagger \wedge \dot{A}_2{}^{0,1} + (\dot{A}_2{}^{0,1})^\dagger \wedge \dot{A}_1{}^{0,1}\right. \\ &\qquad \left. + \dot{\Phi}_1 \wedge (\dot{\Phi}_2)^\dagger + \dot{\Phi}_2 \wedge (\dot{\Phi}_1)^\dagger\right). \end{aligned} \tag{2.83}$$

The L^2-norm of an arbitrary pair $(\dot{A}^{0,1}, \dot{\Phi})$ in $T_{(A,\Phi)}\mathcal{N}$ need not be finite, so this is not an honest metric. Formally, g induces a symplectic form $\omega_I(\cdot,\cdot) = g(I\cdot,\cdot)$, where $I \cdot (\dot{A}^{0,1}, \Phi) = (i\dot{A}^{0,1}, i\Phi)$.

The S^1-action generates a vector field $X = (X_{A^{1,0}}, X_\Phi)$ on $\mathcal{A}^{0,1} \otimes \Omega^{1,0}(\mathbb{CP}^1, \mathrm{End}\ E \otimes \mathbb{C})$. Identify $\mathcal{A}^{0,1}$ with $\Omega^{0,1}(\mathbb{C}, \mathfrak{sl}(E))$ by choosing trivial connection d. Moreover, for convenience, let $\vartheta = \mathrm{Arg}\ z$. Then the infinitesimal S^1-action at $(d + A^{0,1}, \Phi)$ is

$$\begin{aligned} X_\Phi &= \left.\frac{d}{d\theta}\right|_{\theta=0} e^{i\theta}\rho^*\Phi = i\frac{N}{K+N}\Phi - \frac{K}{K+N}\partial_\vartheta\Phi \\ X_{A^{0,1}} &= \left.\frac{d}{d\theta}\right|_{\theta=0} \rho^* A^{0,1} = i\frac{K}{K+N}A^{0,1} - \frac{K}{K+N}\partial_\vartheta A^{0,1} \end{aligned} \tag{2.84}$$

where $\partial_\vartheta : \Omega^{p,q}(\mathbb{CP}^1, \mathrm{End}\ E) \to \Omega^{p,q}(\mathbb{CP}^1, \mathrm{End}\ E)$ denotes the partial derivative with respect to ∂_ϑ. Alternatively, this can more compactly be written as:

$$\begin{aligned} X_\Phi &= -\frac{K}{K+N}\partial_\vartheta\left(\mathrm{e}^{-iN/K\vartheta}\Phi\right)\mathrm{e}^{iN/K\vartheta} \\ X_{A^{0,1}} &= -\frac{K}{K+N}\partial_\vartheta\left(\mathrm{e}^{-i\vartheta}A^{0,1}\right)\mathrm{e}^{i\vartheta} \end{aligned} \tag{2.85}$$

We compute that $\iota_X \omega_I$ is

$$
\begin{aligned}
\left(\iota_X \omega_I\right)\left(\dot{A}^{0,1}, \dot{\Phi}\right) &= \frac{i}{2} \int_{\mathbb{CP}^1} \operatorname{Tr}\left(X_{A^{1,0}} \wedge \dot{A}^{0,1} - \dot{A}^{1,0} \wedge X_{A^{0,1}} + X_{\Phi} \wedge \dot{\Phi}^{\dagger} - \dot{\Phi} \wedge X_{\Phi^{\dagger}}\right) \quad (2.86) \\
&= -\frac{i}{2} \frac{K}{K+N} \int_{\mathbb{CP}^1} \operatorname{Tr}\left(\partial_{\vartheta}\left(\mathrm{e}^{i\vartheta} A^{1,0}\right) \mathrm{e}^{-i\vartheta} \wedge \dot{A}^{0,1} - \dot{A}^{1,0} \wedge \partial_{\vartheta}\left(\mathrm{e}^{-i\vartheta} A^{0,1}\right) \mathrm{e}^{i\vartheta}\right. \\
&\qquad \left. \partial_{\vartheta}\left(\mathrm{e}^{-iN/K\vartheta} \Phi\right) \mathrm{e}^{iN/K\vartheta} \wedge \dot{\Phi}^{\dagger} - \dot{\Phi} \wedge \partial_{\vartheta}\left(\mathrm{e}^{iN/K\vartheta} \Phi^{\dagger}\right) \mathrm{e}^{-iN/K\vartheta}\right),
\end{aligned}
$$

when the integral is finite.

Lemma 2.3.4. *Let $(A, \Phi) \in \mathcal{N}$ and $(\dot{A}^{0,1}, \dot{\Phi})$ be a variation such that*

$$\iota_X \omega_I(\dot{A}^{0,1}, \dot{\Phi}),$$

defined in Eq. 2.86, is well-defined and finite.

Then,

$$\mathrm{d}\mu_3(\dot{A}^{0,1}, \dot{\Phi}) = \iota_X \omega_I(\dot{A}^{0,1}, \dot{\Phi}). \tag{2.87}$$

Proof. Let

$$
\begin{aligned}
\mu(A, \Phi) &= \frac{1}{4(N+K)} \int_{\mathbb{CP}^1} \operatorname{Tr}\left(c_1 \Phi \wedge \Phi^{\dagger} + c_2\left(\Phi \wedge \partial_{\vartheta} \Phi^{\dagger} - \partial_{\vartheta} \Phi \wedge \Phi^{\dagger}\right)\right. \qquad (2.88) \\
&\qquad \left. + c_3\left(-A^{1,0} \wedge A^{0,1}\right) + c_4\left(-A^{1,0} \wedge \partial_{\vartheta} A^{0,1} + \partial_{\vartheta} A^{1,0} \wedge A^{0,1}\right)\right).
\end{aligned}
$$

We will prove that $\mu_3 = \mu$ satisfies Eq. 2.87 for constants

$$c_1 = -2N \qquad c_2 = iK \qquad c_3 = -2K \qquad c_4 = -iK. \tag{2.89}$$

Compute $\mathrm{d}\mu$ at the pair $(A^{0,1}, \Phi)$:

$$\mathrm{d}\mu(\dot{A}^{0,1}, \dot{\Phi}) \tag{2.90}$$

$$
\begin{aligned}
&= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mu(A^{0,1}+\varepsilon\dot{A}^{0,1}, \Phi+\varepsilon\dot{\Phi}) \\
&= \frac{1}{4(N+K)}\int_{\mathbb{CP}^1} \mathrm{Tr}\ c_1\left(\dot{\Phi}\wedge\Phi^\dagger+\Phi\wedge\dot{\Phi}^\dagger\right) \\
&\qquad +c_2\left(\dot{\Phi}\wedge\partial_\vartheta\Phi^\dagger+\Phi\wedge\partial_\vartheta\dot{\Phi}^\dagger-\partial_\vartheta\dot{\Phi}\wedge\Phi^\dagger-\partial_\vartheta\Phi\wedge\dot{\Phi}^\dagger\right) \\
&\qquad +c_3\left(-\dot{A}^{1,0}\wedge A^{0,1}-A^{1,0}\wedge\dot{A}^{0,1}\right) \\
&\qquad +c_4\Big(-\dot{A}^{1,0}\wedge\partial_\vartheta A^{0,1}-A^{1,0}\wedge\partial_\vartheta\dot{A}^{0,1}+ \\
&\qquad\qquad \partial_\vartheta\dot{A}^{1,0}\wedge A^{0,1}+\partial_\vartheta A^{1,0}\wedge\dot{A}^{0,1}\Big) \\
&= \frac{1}{4(N+K)}\int_{\mathbb{CP}^1} \mathrm{Tr}\ c_1\left(\dot{\Phi}\wedge\Phi^\dagger+\Phi\wedge\dot{\Phi}^\dagger\right)+2c_2\left(\dot{\Phi}\wedge\partial_\vartheta\Phi^\dagger-\partial_\vartheta\Phi\wedge\dot{\Phi}^\dagger\right) \\
&\qquad +c_3\left(-\dot{A}^{1,0}\wedge A^{0,1}-A^{1,0}\wedge\dot{A}^{0,1}\right) \\
&\qquad +2c_4\left(-\dot{A}^{1,0}\wedge\partial_\vartheta A^{0,1}+\partial_\vartheta A^{1,0}\wedge\dot{A}^{0,1}\right) \\
&= \frac{1}{4(N+K)}\int_{\mathbb{CP}^1} \mathrm{Tr}\ \dot{\Phi}\wedge(c_1\Phi^\dagger+2c_2\partial_\vartheta\Phi^\dagger)+(c_1\Phi-2c_2\partial_\vartheta\Phi)\wedge\dot{\Phi}^\dagger \\
&\qquad +\dot{A}^{1,0}\wedge(-c_3A^{0,1}-2c_4\partial_\vartheta A^{0,1})+(-c_3A^{1,0}+2c_4\partial_\vartheta A^{1,0})\wedge\dot{A}^{0,1}
\end{aligned}
$$

Then, comparing with Eq. 2.86, see that the constants c_1, c_2, c_3, and c_4 are correct. In particular, looking at the $\dot{\Phi}^\dagger$ terms:

$$
-\frac{i}{2}\frac{K}{K+N}\left(\partial_\vartheta\left(\mathrm{e}^{-iN/K\vartheta}\Phi\right)\mathrm{e}^{iN/K\vartheta}\right) = \frac{(-2N)}{4(K+N)}\Phi-2\frac{(iK)}{4(K+N)}\partial_\vartheta\Phi, \tag{2.91}
$$

hence $c_1=-2N$ and $c_2=iK$. Looking at the $\dot{A}^{1,0}$ terms:

$$
\frac{i}{2}\frac{K}{K+N}\partial_\vartheta\left(\mathrm{e}^{-i\vartheta}A^{0,1}\right)\mathrm{e}^{i\vartheta} = -\frac{(-2K)}{4(K+N)}A^{0,1}-2\frac{(-iK)}{4(K+N)}\partial_\vartheta A^{0,1}, \tag{2.92}
$$

hence $c_3=-2K$ and $c_4=-iK$. □

Proposition 2.3.5. *Given* **b**, *an ordered* K*-partition of* N, *let* (A,Φ) *be the associ-*

ated S^1-fixed point. Then, for μ_3 defined in Eq. 2.81,

$$\mu_3(A, \Phi) = \frac{-i\pi K}{K+N} \|B\mathbf{b}\|^2. \tag{2.93}$$

Proof. For (A, Φ) as in Theorem 2.2.4,

$$\begin{aligned} \partial_\vartheta \Phi &= i\Phi \odot \begin{pmatrix} & b_1 & & \\ & & \cdots & \\ & & & b_{K-1} \\ b_K & & & \end{pmatrix} \\ \partial_\vartheta A^{0,1} &= iA^{0,1} \end{aligned} \tag{2.94}$$

where $\odot$ is the binary operation of taking element-wise multiplication of two matrices, i.e. $(A \odot B)_{ij} = A_{ij} \cdot B_{ij}$. Then, the terms of μ involving Φ simplify:

$$\begin{aligned} & c_1 \Phi\Phi^\dagger + c_2 \left(\Phi \partial_\vartheta \Phi^\dagger - \partial_\vartheta \Phi \Phi^\dagger\right) \\ = \; & \Phi\Phi^\dagger \left(c_1 \mathrm{Id} - 2ic_2 \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_K \end{pmatrix} \right) \\ = \; & 2K\Phi\Phi^\dagger \left(\begin{pmatrix} b_1 - \frac{N}{K} & & \\ & b_2 - \frac{N}{K} & \\ & & b_K - \frac{N}{K} \end{pmatrix} \right) \end{aligned} \tag{2.95}$$

Together, the terms of μ involving A vanish. Consequently,

$$\begin{aligned} \mu_3(A^{0,1}, \Phi) &= \frac{2K}{4(N+K)} \int_{\mathbb{CP}^1} \mathrm{Tr}\, \Phi\Phi^\dagger \begin{pmatrix} b_1 - \frac{N}{K} & & & \\ & b_2 - \frac{N}{K} & & \\ & & \ddots & \\ & & & b_K - \frac{N}{K} \end{pmatrix} \\ &\overset{Eq.\ 2.55}{=} \frac{2K}{4(N+K)} \int_{\mathbb{CP}^1} \sum_{i=1}^{K} |z|^{\frac{2N}{K}} \mathrm{e}^{u_i - u_{i+1}} \left(b_i - \frac{N}{K}\right) \mathrm{d}z\mathrm{d}\bar{z} \end{aligned} \tag{2.96}$$

$$
\begin{aligned}
&\overset{Eq.\,2.13}{=} && \frac{1}{2}\int_{\mathbb{CP}^1}|z|^{\frac{2N}{K}}\sum_{i=1}^{K}\mathrm{e}^{u_i-u_{i+1}}(c_{i+1}-c_i)\mathrm{d}z\mathrm{d}\bar{z} \\
&= && \frac{1}{2}\int_{\mathbb{CP}^1}|z|^{\frac{2N}{K}}\sum_{i=1}^{K}c_i\left(\mathrm{e}^{u_{i-1}-u_i}-\mathrm{e}^{u_i-u_{i+1}}\right)\mathrm{d}z\mathrm{d}\bar{z} \\
&\overset{Eq.2.14}{=} && \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\sum_{i=1}^{K}c_i\left(\frac{1}{4}\frac{1}{|z|}\frac{d}{d|z|}\left(|z|\frac{du_i}{d|z|}\right)\right)(-2i|z|\mathrm{d}|z|\mathrm{d}\vartheta) \\
&= && \frac{(2\pi)(-2i)}{(2)(4)}\left[\sum_{i=1}^{K}c_i\left(|z|\frac{du_i}{d|z|}\right)\right|_0^{\infty} \\
&\overset{Eq.2.16,2.17}{=} && \frac{(2\pi)(-2i)}{(2)(4)}\sum_{i=1}^{K}c_i\left(0-\left(-\frac{2(K+N)}{K}c_i\right)\right) \\
&= && \frac{-i\pi(K+N)}{K}\|\mathbf{c}\|^2 \\
&\overset{Eq.2.19}{=} && \frac{-i\pi K}{K+N}\|B\mathbf{b}\|^2
\end{aligned}
$$

□

2.3.4 Relation between μ_i

Let $\mathbf{b}$ be an ordered K-partition of N. From Theorem 2.2.4, there is an associated S^1-fixed point (A,Φ). In the previous three subsections, we defined and computed three different quantities: μ_1,μ_2,μ_3 at (A,Φ). All three are scalar multiples of $\|B\mathbf{b}\|^2$:

$$
\begin{aligned}
\mu_1 &= \frac{\pi K}{K+N}\|B\mathbf{b}\|^2 && \text{(See Eq. 2.66, 2.69.)} \\
\mu_2 &= \|B\mathbf{b}\|^2 && \text{(See Eq. 2.79, 2.80.)} \\
\mu_3 &= \frac{-i\pi K}{K+N}\|B\mathbf{b}\|^2 && \text{(See Eq. 2.81, 2.93.)}
\end{aligned}
$$

Consequently,

Proposition 2.3.6. *Given* $\mathbf{b}$*, an ordered* K*-partition of* N*, let* (A, Φ) *be the associated* S^1*-fixed point. Define*

$$\mu := \frac{K}{K+N}\|B\mathbf{b}\|^2. \tag{2.97}$$

Then,

$$\mu = \frac{1}{\pi}\mu_1 = \frac{K}{K+N}\mu_2 = \frac{i}{\pi}\mu_3 = \frac{K+N}{K}\|\mathbf{c}\|^2. \tag{2.98}$$

2.4 $\mathcal{W}_K$-algebra minimal models

In this penultimate section, we review some basic facts about $\mathcal{W}$-algebras. In the last section, we will relate solutions of Hitchin's equations fixed by the S^1-action to certain $\mathcal{W}$-algebra representations.

Two-dimensional quantum field theories with conformal symmetry are particularly nice since they can be solved exactly. $\mathcal{W}$-algebras are extended symmetry algebras. $\mathcal{W}$-algebras are labeled by an integer $K \geq 2$. We will include the label K as a subscript $\mathcal{W}_K$. For $K = 2$, the $\mathcal{W}_2$-algebra is the Virasoro algebra.

We are interested in representations of $\mathcal{W}_K$. In this section we consider certain $\mathcal{W}_K$ algebra minimal models. At the crudest level of sets, $\mathcal{W}_K$-algebra minimal models of are certain collections of representations of the $\mathcal{W}_K$-algebra. $\mathcal{W}_K$-algebra minimal models of are classified by a pair of integers. Of all such pairs, we only consider pairs $(K, K+N)$. In §2.4.1 we review the classification of irreducible representations of $(K, K+N)$ $\mathcal{W}_K$-algebra minimal model. In §2.4.2, we assign a number—the effective central charge—to each irreducible representation. In Theorem 2.5.1, we state the relation between the effective central charge (§2.4.2) and the number μ we associated to a solution of Hitchin's equations fixed by the S^1-action(§2.3).

2.4.1 Irreducible representations of $\mathcal{W}_K$

As a set, the $(K, K+N)$-minimal model consists of representations of the $\mathcal{W}_K$-algebra. Irreducible representations are labeled by highest weights. In turn, the highest weights are classified by cyclic K-partitions of N (Definition 2.2.2). It will be convenient to use a different classification data:

Definition 2.4.1. An (K, N) **non-increasing sequence** is a K-tuple of integers $\mathbf{n} = (n_1, \cdots, n_K)$ such that $N \geq n_1 \geq n_2 \geq n_K = 0$.

The following map, ψ, gives the obvious bijection between the set of ordered K-partitions of N and the set of (K, N) non-increasing sequences, indicated in Figure 2.2

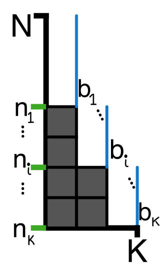

Figure 2.2: *To each non-increasing sequence, we may associate a Young diagram, where the height of the j^{th} column is n_j. The bijection in Eq. 2.99 between the set of ordered K-partitions of N and the set of (K, N) non-increasing sequences is the obvious one.*

$$\psi : \ \{\text{ordered } K\text{-partitions of } N\} \ \to \{(N, K) \text{ non-increasing sequences}\} \tag{2.99}$$

$$\mathbf{b} = (b_1, \cdots, b_K) \qquad \mapsto \mathbf{n} = (n_1, \cdots, n_K)$$
$$\text{where } n_i = N - \textstyle\sum_{j=1}^{i} b_j.$$

The inverse map is $\psi^{-1}(\mathbf{n}) = \mathbf{b}$ where $b_1 = N - n_1$ and $b_i = n_{i-1} - n_i$ for $i \neq 1$.$\triangleleft$

Proposition 2.4.1 ([BS92, CS15]). *Highest weights of the* $(K, K+N)$ $\mathcal{W}_K$*-algebra minimal model are classified by cyclic* K*-partitions of* N*. Concretely, given* $\mathbf{b}$*, an ordered* K*-partition of* N *in the equivalence class* $[\mathbf{b}]$*, the associated highest weight is*

$$\Lambda = P_{\mathbf{1}^\perp} \psi(\mathbf{b}) \tag{2.100}$$

where $P_{\mathbf{1}^\perp}$ *denotes orthogonal projection onto* $\mathbf{1}^\perp \subset \mathbb{R}^K$ *where* $\mathbf{1} = (1, \cdots, 1)$*.*

Proof. See [BS92] Eq. 6.73 for classification. The result is also stated in Eq. 4.71 of [CS15]. $\square$

2.4.2 An associated co-character

In this section, we associate a number—the effective central charge—to the irreducible representation with highest weight $\Lambda = P_{\mathbf{1}^\perp}\mathbf{n}$, where $\mathbf{n}$ is an (K, N) non-increasing sequence.

Proposition 2.4.2 ([BS92]). *Given a* (K, N) *non-increasing sequence* $\mathbf{n}$ *the effective central charge of the representation associated to highest weight* $\Lambda = P_{\mathbf{1}^\perp}\mathbf{n}$ *is*

$$c_{\text{eff}}(\mathbf{n}) = K - 1 - \frac{12K}{K+N} \left\| P_{\mathbf{1}^\perp}\mathbf{n} - \frac{N}{K}\rho \right\|^2 \tag{2.101}$$

where

$$\rho = \frac{1}{2}\left(K-1, K-3, \cdots, 3-K, 1-K\right) \tag{2.102}$$

is $\frac{1}{2}$ *of the sum of the positive weights.*

2.5 Main Theorem

Given $\mathbf{b}$, an ordered K-partition of N, we associated

- (A, Φ), a S^1-fixed point (Theorem 2.2.4), and a number μ (Proposition 2.3.6), and
- Λ, a highest weight in the $(K, K+N)$ minimal model of $\mathcal{W}_K$ (Proposition 2.4.1), and a number c_{eff} (Proposition 2.4.2).

These two numbers are

$$\begin{aligned} \mu &= \frac{K}{K+N} \|B\mathbf{b}\|^2 \\ c_{\text{eff}} &= K - 1 - \frac{12K}{K+N} \left\| P_{\mathbf{1}}^{\perp}\mathbf{n} - \frac{N}{K}\rho \right\|^2 \end{aligned} \tag{2.103}$$

where $P_{\mathbf{1}}^{\perp}$ is the orthogonal projection onto the subspace orthogonal to $\mathbf{1}$, ρ is as in Eq. 2.102, B is as in equation 2.20. Then,

Theorem 2.5.1. *Let* $\mathbf{b}$ *be an ordered K-partition of N. Let Λ be the height weight in the $(K, K+N)$ $\mathcal{W}_K$-algebra minimal model associated to the cyclic ordered partition* $[\mathbf{b}]$. *Let (A, Φ) by the associated S^1 fixed point. Then*

$$\mu = \frac{1}{12}\left(K - 1 - c_{\text{eff}}\right). \tag{2.104}$$

Remark 2.5.1. ▷ As possible explanation for the constant $\frac{1}{12}$ appearing in Eq. 2.104, note that this $\frac{1}{12}$ appears in the usual algebra relations merely as a matter of convention. For example, for $K = 2$, the $\mathcal{W}_2$-algebra is the Virasoro algebra. The Virasoro algebra is spanned by elements $\{L_m\}_{m\in\mathbb{Z}}$ and central element c. The relations are $[c, L_m] = 0$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{2.105}$$

Notice the "$\frac{1}{12}$" appearing.◁

Proof. Using Eq. 2.103, note that

$$\frac{1}{12}(K - 1 - c_{\text{eff}}) = \frac{K}{K+N}\left\| P_{\mathbf{1}}^{\perp}\mathbf{n} - \frac{N}{K}\rho \right\|^2. \tag{2.106}$$

To prove Eq. 2.104 holds, we prove something stronger. Rather than proving that the norms of the two vectors $P_{\mathbf{1}}^{\perp}\mathbf{n} - \frac{N}{K}\rho$ and $B\mathbf{b}$ are equal, we prove that the vectors are related by

$$\left(P_{\mathbf{1}}^{\perp}\mathbf{n} - \frac{N}{K}\rho\right) = -MB\mathbf{b} \tag{2.107}$$

where M is the permutation matrix corresponding to permutation $(1\ 2\ \cdots\ K)$, i.e. the (i,j) entry of M is $M_{ij} = \delta_{i+1,j}$ for $i, j \in \mathbb{Z}/K\mathbb{Z}$.

To prove Eq. 2.107, we use the defining property of the matrix B. The matrix B in Eq. 2.20 is the unique matrix such that

$$B(M - \mathrm{Id}) = (M - \mathrm{Id})B = P_{\mathbf{1}^{\perp}}. \tag{2.108}$$

and $B\mathbf{1} = \mathbf{0}$. Additionally, note:

- From Eq. 2.108

$$MB = BM. \tag{2.109}$$

- The (K,N) non-increasing sequence $\mathbf{n}$ is related to $\mathbf{b}$ by $\mathbf{n} = \psi(\mathbf{b})$ in Eq. 2.99. Alternatively, this can be expressed

$$\mathbf{b} = (M^{-1} - \mathrm{Id})\mathbf{n} + \begin{pmatrix} N \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.110}$$

(The last term above appears because b_1 is equal to $N - n_1$ rather than $n_K - n_1 = 0 - n_1$.)

- Lastly,

$$BM \begin{pmatrix} N \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{N}{K}\rho. \tag{2.111}$$

Consequently,we can now prove that the relevant vectors satisfy Eq. 2.107.

$$\begin{aligned} -MB\mathbf{b} &= -MB\left((M^{-1} - \mathrm{Id})\mathbf{n} + \begin{pmatrix} N \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) && (2.112) \\ &= (M - \mathrm{Id})B\mathbf{n} - BM \begin{pmatrix} N \\ 0 \\ \vdots \\ 0 \end{pmatrix} && (2.113) \\ &= P_{\mathbf{1}^\perp}\mathbf{n} - \frac{N}{K}\rho. && (2.114) \end{aligned}$$

□

Chapter 3

Opers versus nonabelian Hodge

This is joint work with Olivia Dumitrescu, Georgios Kydonakis, Rafe Mazzeo, Motohico Mulase, and Andy Neitkze.

We thank the American Institute of Mathematics for its hospitality during the workshop "New perspectives on spectral data for Higgs bundles," where this work was initiated. We also thank the organizers and all the participants of the workshop, and particularly Philip Boalch for posing the question which led to this work.

3.1 Introduction

3.1.1 Summary

This paper proves an extension of a conjecture formulated by Gaiotto in [Gai14], Conjecture 3.3.1 below. The theorem concerns a simple complex Lie group G; the case $G = SL(K, \mathbb{C})$ is Theorem 3.3.2 below.

3.1.2 The case of $G = SL(2, \mathbb{C})$

The fundamental example is $G = SL(2, \mathbb{C})$. Suppose we are given a compact Riemann surface C of genus $g \geq 2$ and a holomorphic quadratic differential ϕ_2 on C. This data determines two natural families of $SL(2, \mathbb{C})$-connections, as follows.

- First, we consider the family of *opers* determined by ϕ_2. These are a global version of the locally-defined second-order differential operator (Schrödinger operator)

$$\mathcal{D}_\hbar : \psi \mapsto \left[\hbar^2 \partial_z^2 + P_2(z)\right] \psi(z), \tag{3.1}$$

where $\hbar \in \mathbb{C}^\times$ and $\phi_2 = P_2(z)\mathrm{d}z^2$ locally. The operator $\mathcal{D}_\hbar$ makes sense globally with the following two stipulations:

  - We consider ψ as a section of $K_C^{-1/2}$,
  - We use only coordinate charts in the atlas on C coming from Fuchsian uniformization, so that the transition maps are Möbius transformations. In other coordinate systems $\mathcal{D}_\hbar$ would take a more complicated form.

  By a standard maneuver, replacing ψ by its 1-jet $\begin{pmatrix} -\hbar^2\psi' \\ \hbar\psi \end{pmatrix}$, we can convert $\mathcal{D}_\hbar$ to a flat connection $\nabla_{\hbar,\phi_2}$ in a certain rank 2 vector bundle $E_\hbar$ over C. Holomorphically, $E_\hbar$ is an extension of $K_C^{-1/2}$ by $K_C^{1/2}$. $E_\hbar$ has distinguished local trivializations defined canonically in terms of coordinate charts on C, and in such a trivialization,

$$\nabla_{\hbar,\phi_2} = \mathrm{d} + \hbar^{-1} \begin{pmatrix} 0 & 1 \\ P_2 & 0 \end{pmatrix}, \qquad \hbar \in \mathbb{C}^\times. \tag{3.2}$$

- We also consider a *Higgs bundle* determined by ϕ_2: this is the bundle $E = K_C^{1/2} \oplus K_C^{-1/2}$, equipped with its standard holomorphic structure $\bar\partial_E$, and a "Higgs field" $\varphi \in \Omega^{1,0}(\mathrm{End}\, E)$ represented in local trivializations by

$$\varphi = \begin{pmatrix} 0 & 1 \\ P_2 & 0 \end{pmatrix}. \tag{3.3}$$

According to the nonabelian Hodge theorem (Theorem 3.2.2), associated to $(E, \bar{\partial}_E, \varphi)$ there is a canonical family of flat connections in E, of the form

$$\nabla_{\zeta,\phi_2} = \zeta^{-1}\varphi + D + \zeta\varphi^{\dagger_h}, \qquad \zeta \in \mathbb{C}^\times \tag{3.4}$$

where the Hermitian metric h is determined by solving a certain elliptic PDE on C (Hitchin's equation, (3.6) below), and D is the associated Chern connection.

The families (3.2), (3.4) appear similar in certain respects. Indeed, their leading terms in the $\hbar \to 0$ or $\zeta \to 0$ limit are the same if we set $\hbar = \zeta$. However, these two families are *not* exactly the same.

Gaiotto in [Gai14] proposed that the relation between them should be as follows. Introduce an additional parameter $R \in \mathbb{R}_+$ and rescale the Higgs field by $\varphi \to R\varphi$; this leads to a 2-parameter analogue of (3.4),

$$\nabla_{\zeta,R,\phi_2} = R\zeta^{-1}\varphi + D(R) + R\zeta\varphi^{\dagger_{h(R)}}, \qquad \zeta \in \mathbb{C}^\times, R \in \mathbb{R}_+. \tag{3.5}$$

Now consider a scaling limit of ∇_{ζ,R,ϕ_2} where $\zeta = R\hbar$ and $R \searrow 0$. Gaiotto proposed that this limit should exist and be equivalent to $\nabla_{\hbar,\phi_2}$. In Section 3.3.2 below we prove that this is indeed the case.

3.1.3 The case of $G = SL(K, \mathbb{C})$

The story just described has an extension where order-2 differential operators, quadratic differentials ϕ_2 and $SL(2, \mathbb{C})$-connections are replaced by order-K differential operators, tuples $(\phi_2, \ldots, \phi_K)$ of holomorphic differentials, and $SL(K, \mathbb{C})$-connections respectively. We treat this extension in Section 3.3.3.

3.2 Background, for $G = SL(K, \mathbb{C})$

In this section we give some background on the main players in our story: Hitchin's equations, the Hitchin section, and opers. We specialize to the case $G = SL(K, \mathbb{C})$ and thus work with vector bundles rather than principal bundles. The only parts which may not be completely standard are the last two, §3.2.10 and §3.2.11; in these sections we describe a concrete construction of $SL(K, \mathbb{C})$-opers, and its relation to the fact that opers are K-th order scalar differential operators.

3.2.1 Hitchin's equations

Fix a compact Riemann surface C of genus $g \geq 2$ and an integer $K \geq 2$. We consider tuples (E, h, D, φ) comprised of:

- A rank K complex vector bundle E over C, equipped with a trivialization of $\det E$,
- A Hermitian metric h in E which induces the trivial metric on $\det E$,
- An h-unitary connection D in E,
- A traceless section φ of $\mathrm{End}(E) \otimes K_C$.

Hitchin's equations [Hit87] are a system of nonlinear PDE for these data:

$$F_D + [\varphi, \varphi^{\dagger_h}] = 0, \tag{3.6a}$$

$$\bar{\partial}_D \varphi = 0. \tag{3.6b}$$

Here F_D denotes the curvature of D, $\dagger_h$ means the adjoint with respect to the metric h, and $\bar{\partial}_D$ is the $(0,1)$ part of the connection D.

We shall actually be considering a rescaled version of (3.6),

$$F_D + R^2[\varphi, \varphi^{\dagger_h}] = 0, \tag{3.7a}$$

$$\bar{\partial}_D \varphi = 0, \tag{3.7b}$$

obtained by replacing $\varphi \to R\varphi$, where $R \in \mathbb{R}^+$.

3.2.2 Higgs bundles

Now suppose given a solution (E, h, D, φ) of (3.7). The operator $\bar{\partial}_D$ gives a holomorphic structure on E. Equation (3.7b) then says that φ is a holomorphic section of $\mathrm{End}(E) \otimes K_C$. Thus the tuple $(E, \bar{\partial}_D, \varphi)$ is an $SL(K, \mathbb{C})$-Higgs bundle:

Definition 3.2.1. An **$SL(K, \mathbb{C})$-Higgs bundle over C** is a tuple $(E, \bar{\partial}, \varphi)$:

- A rank K complex vector bundle E over C, equipped with a trivialization of the determinant bundle $\det E$,
- A holomorphic structure $\bar{\partial}$ on E,
- A traceless holomorphic section φ of $\mathrm{End}(E) \otimes K_C$.

3.2.3 Harmonic metrics

Conversely, suppose given an $SL(K, \mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$ and a Hermitian metric h on E inducing the trivial metric on $\det E$. Then there is a unique

h-unitary connection D_h in E with $\bar{\partial}_{D_h} = \bar{\partial}$ (Chern connection). The equation (3.7b) automatically holds when $D = D_h$. The equation (3.7a) with $D = D_h$ becomes a nonlinear PDE for the metric h; we say h is a *harmonic metric* if it solves this equation:

Definition 3.2.2. Given an $SL(K,\mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$, and $R \in \mathbb{R}^+$, a *harmonic metric with parameter* R is a Hermitian metric h on E, inducing the trivial metric on $\det E$, such that

$$F_{D_h} + R^2[\varphi, \varphi^{\dagger_h}] = 0. \tag{3.8}$$

Thus, we have

Proposition 3.2.1. *Given an $SL(K,\mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$, $R \in \mathbb{R}^+$, and a harmonic metric h with parameter R, the tuple (E, h, D_h, φ) gives a solution of Hitchin's equations* (3.7).

Definition 3.2.3. An $SL(K,\mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$ is called *stable* if there is no holomorphic subbundle $E' \subset E$ such that $\varphi(E') \subset E' \otimes K_C$ and $\deg(E') > 0$.

The following key result, sometimes called the "nonabelian Hodge theorem," is proven in [Sim88]:[1]

Theorem 3.2.2. *Given a stable $SL(K,\mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$, and any $R \in \mathbb{R}^+$, there exists a unique harmonic metric h with parameter R.*

[1] More precisely, the theorem in [Sim88] concerns $GL(K)$-bundles rather than $SL(K,\mathbb{C})$-bundles, but it is straightforward to deduce the version for $SL(K,\mathbb{C})$-bundles.

Combining this with Proposition 3.2.1, we see that given a stable Higgs bundle and a parameter R, we obtain a solution of Hitchin's equations (3.7) with parameter R.

3.2.4 Real twistor lines

Given a solution (E, h, D, φ) of Hitchin's equations (3.7) with parameter R, there is a corresponding family of flat non-unitary connections in E, given by the formula

$$\nabla_\zeta = \zeta^{-1} R\varphi + D + \zeta R \varphi^{\dagger_h}, \qquad \zeta \in \mathbb{C}^\times \tag{3.9}$$

Indeed, the statement that ∇_ζ is flat for all $\zeta \in \mathbb{C}^\times$ is equivalent to (3.7). The family (3.9) is sometimes called the "real twistor line" corresponding to the Higgs bundle $(E, \bar{\partial}_E, \varphi)$.

3.2.5 A canonical $\mathfrak{sl}_2$-triple

To make the next definition we need some preliminary notations. First we define

$$H = \begin{pmatrix} K-1 & & & & \\ & K-3 & & & \\ & & \ddots & & \\ & & & -K+3 & \\ & & & & -K+1 \end{pmatrix}, \tag{3.10}$$

$$X_+ = \begin{pmatrix} 0 & \sqrt{r_1} & & & \\ & 0 & \sqrt{r_2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \sqrt{r_{K-1}} \\ & & & & 0 \end{pmatrix},$$

$$X_- \;=\; \begin{pmatrix} 0 & & & & \\ \sqrt{r_1} & 0 & & & \\ & \sqrt{r_2} & \ddots & & \\ & & \ddots & 0 & \\ & & & \sqrt{r_{K-1}} & 0 \end{pmatrix},$$

where

$$r_i = i(K-i). \tag{3.11}$$

These make up an $\mathfrak{sl}_2$-triple:

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = H. \tag{3.12}$$

In addition, for $n \geq 1$, choose (once and for all) a nonzero real matrix X_n, such that only the ij entries with $j - i = n$ are nonzero (the n^{th} super-diagonal), or equivalently

$$[H, X_n] = 2nX_n, \tag{3.13}$$

and also

$$[X_+, X_n] = 0. \tag{3.14}$$

For example, when $K = 4$ we can choose

$$X_1 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

For later use we record a few facts, obtained by direct computation:

Proposition 3.2.3. *We have the following:*

- *The equations* (3.13), (3.14) *determine* X_n *up to a real constant multiple for* $n > 0$.

- *The equations* (3.13), (3.14) *determine* X_0 *to be a multiple of the identity.*

- *The equations* (3.13), (3.14) *have only the solution* $X_n = 0$ *for* $n < 0$.

- *The solution* X_n *has the antidiagonal symmetry* $(X_n)_{ij} = (X_n)_{K+1-j,K+1-i}$.

3.2.6 The Hitchin component

Definition 3.2.4. The *Hitchin base* is the vector space

$$\mathcal{B} = \bigoplus_{n=2}^{K} H^0(C, K_C^n). \tag{3.16}$$

We denote points of $\mathcal{B}$ by

$$\mathbf{u} = (\phi_2, \ldots, \phi_K). \tag{3.17}$$

Now fix a spin structure on C, i.e. a holomorphic line bundle $\mathcal{L}$ over C equipped with an isomorphism $\mathcal{L}^2 \simeq K_C$. Over each local coordinate chart (U, z) on C, $\mathcal{L}$ has two distinguished trivializations corresponding to the two square roots $\sqrt{\mathrm{d}z}$; we choose one of these arbitrarily for each chart. Then the transition map for $\mathcal{L}$ between charts (U, z) and (U', z') is of the form $(z, s) \sim (z', s' = \alpha_{z,z'} s)$, where

$$\alpha_{z,z'}^2 = \frac{\mathrm{d}z}{\mathrm{d}z'} \tag{3.18}$$

and the choice of square root is related to our choices of $\sqrt{\mathrm{d}z}$ above.

Definition 3.2.5. The *Hitchin component* is a set of stable $SL(K,\mathbb{C})$-Higgs bundles $(E, \bar{\partial}_E, \varphi_{\mathbf{u}})$, parameterized by $\mathbf{u} \in \mathcal{B}$, as follows:

- E is the smooth vector bundle

$$E = \mathcal{L}^{K-1} \oplus \mathcal{L}^{K-3} \oplus \cdots \oplus \mathcal{L}^{-K+3} \oplus \mathcal{L}^{-K+1}. \tag{3.19}$$

  Our distinguished local trivializations of $\mathcal{L}$ induce distinguished local trivializations of E. Note that the exponents appearing in (3.19) are also the diagonal entries of the matrix H. Thus the transition maps between distinguished trivializations of E are

$$\alpha^H_{z,z'} = \begin{pmatrix} \alpha^{K-1}_{z,z'} & & & & \\ & \alpha^{K-3}_{z,z'} & & & \\ & & \ddots & & \\ & & & \alpha^{-K+3}_{z,z'} & \\ & & & & \alpha^{-K+1}_{z,z'} \end{pmatrix}. \tag{3.20}$$

- $\bar{\partial}_E$ is the holomorphic structure on E induced from the one on $\mathcal{L}$.

- Fix a coordinate chart (U, z) and write $\phi_n = P_{n,z} \mathrm{d}z^n$. Then the Higgs field $\varphi_{\mathbf{u}} \in \operatorname{End} E \otimes K_C$ is, relative to the distinguished local trivialization of E,

$$\varphi_{\mathbf{u},z} = \left(X_- + \sum_{n=1}^{K-1} P_{n+1,z} X_n \right) \mathrm{d}z. \tag{3.21}$$

  (Note that this indeed makes global sense, i.e. $\alpha^H_{z,z'} \varphi_{\mathbf{u},z} \alpha^H_{z',z} = \varphi_{\mathbf{u},z'}$.)

Example 3.2.4. *For $K = 5$, the Higgs field $\varphi_{\mathbf{u}}$ is*

$$\varphi_{\mathbf{u}} = \begin{pmatrix} 0 & 2P_2 & 2P_3 & P_4 & P_5 \\ 2 & 0 & \sqrt{6}P_2 & \sqrt{6}P_3 & P_4 \\ & \sqrt{6} & 0 & \sqrt{6}P_2 & 2P_3 \\ & & \sqrt{6} & 0 & 2P_2 \\ & & & 2 & 0 \end{pmatrix} \mathrm{d}z. \tag{3.22}$$

(Here and below, when we are working within a single coordinate chart (U, z), we sometimes drop the explicit subscripts z to reduce clutter.) Note that the characteristic polynomial of this matrix is

$$t^5 - 20P_2t^3 - 14\sqrt{6}P_3t^2 - (24P_4 - 64P_2^2)t - (24P_5 - 32\sqrt{6}P_2P_3), \tag{3.23}$$

so with our conventions, the P_n are not *the coefficients of the characteristic polynomial, but both determine and can be recovered from these coefficients.*

When K is even, the Hitchin component depends on the spin structure which we chose. When K is odd, only even powers of $\mathcal{L}$ appear, so the Hitchin component actually does not depend on the spin structure.

3.2.7 The bilinear pairing

The bundle E given by (3.19) has a nondegenerate complex bilinear pairing Q, coming from the fact that $\mathcal{L}^{-n} = (\mathcal{L}^n)^*$. In our distinguished trivializations this is simply

$$Q = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}. \tag{3.24}$$

The antidiagonal symmetry of $X_\pm$ and the X_n can be restated as saying that they are self-adjoint with respect to Q, i.e. in a local trivialization

$$QX_n^TQ^{-1} = X_n \tag{3.25}$$

and similarly for $X_\pm$. Thus, for any $\mathbf{u} \in \mathcal{B}$, the Higgs field $\varphi_\mathbf{u}$ is also Q-self-adjoint,

$$Q\varphi_\mathbf{u}^TQ^{-1} = \varphi_\mathbf{u}. \tag{3.26}$$

We define $\mathrm{End}_Q E$ to be the subalgebra of traceless Q-skew-adjoint endomorphisms, i.e. $\chi \in \mathrm{End}_Q E$ means

$$Q\chi^T Q^{-1} = -\chi, \qquad \mathrm{Tr} Q = 0. \tag{3.27}$$

We then have

Lemma 3.2.5. *If $\chi \in \mathrm{End}_Q E$, then $[X_+, \chi] = 0$ if and only if $\chi = 0$.*

Proof. This follows directly from Proposition 3.2.3, which says that if $[X_+, \chi] = 0$, then χ is a combination of $X_0, X_1, \ldots, X_K$, and then is Q-self-adjoint. $\square$

3.2.8 The natural metric

The Higgs bundle corresponding to the origin of $\mathcal{B}$ is particularly important:

Definition 3.2.6. The *uniformizing Higgs bundle* is the element $(E, \bar{\partial}_E, \varphi_{\mathbf{0}})$ of the Hitchin component, where $\mathbf{0} = (0, 0, \ldots, 0) \in \mathcal{B}$.

Here is the reason for the name. By the uniformization theorem, the conformal class given by the complex structure on C contains a unique Riemannian metric $g_\natural$ with constant curvature -4. More generally, $g_\natural/R^2$ is the unique metric with constant curvature $-4R^2$. This in turn induces a metric on E, as follows:

Definition 3.2.7. The *natural metric* $h_\natural(R)$ on the bundle E of (3.19) is determined as follows. It is orthogonal with respect to the decomposition (3.19), and on $\mathcal{L}^n \subset E$, it is induced by $g_\natural/R^2$, i.e.,

$$h_\natural(R) = R^n g_\natural^{-n/2} \qquad \text{on } \mathcal{L}^n \subset E. \tag{3.28}$$

We write $h_\natural$ for this metric when $R = 1$, or equivalently,

$$h_\natural(R) = h_\natural \tau_R \tag{3.29}$$

where $\tau_R : E \to E$ rescales $\mathcal{L}^n \subset E$ by the factor R^n.

We also describe $h_\natural(R)$ relative to the distinguished local trivializations of E. In a local coordinate chart (U, z),

$$g_\natural = \lambda_{\natural,z}^2 \mathrm{d}z\mathrm{d}\bar{z}, \quad \text{where} \quad \partial_{\bar{z}}\partial_z \log \lambda_{\natural,z} - \lambda_{\natural,z}^2 = 0. \tag{3.30}$$

Then

$$h_{\natural,z}(R) = R^H \lambda_{\natural,z}^{-H}, \tag{3.31}$$

relative to the distinguished local trivialization over (U, z).

The next proposition, from [Hit92], explains the importance of $h_\natural(R)$ in this story.

Proposition 3.2.6. *The harmonic metric on the uniformizing Higgs bundle* $(E, \bar{\partial}_E, \varphi_{\mathbf{0}})$ *with parameter* R *is* $h_\natural(R)$.

Proof. We just compute directly in the distinguished trivializations:

$$\begin{aligned} F_{D_{h_\natural(R)}} + R^2 \left[\varphi_{\mathbf{0}}, \varphi_{\mathbf{0}}^{\dagger_{h_\natural(R)}}\right] &= \left(\partial_{\bar{z}}\partial_z \log(\lambda_\natural) H + \left[X_-, \lambda_\natural^2 X_+\right]\right) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \left(\partial_{\bar{z}}\partial_z \log(\lambda_\natural) - \lambda_\natural^2\right) H \,\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 0. \end{aligned} \tag{3.32}$$

□

3.2.9 $SL(K,\mathbb{C})$-opers

We now recall the notion of $SL(K,\mathbb{C})$-*oper*:

Definition 3.2.8. An $SL(K,\mathbb{C})$-oper on C is a tuple $(E,\nabla,F_\bullet)$:

- A rank K complex vector bundle E over C, equipped with a trivialization of the determinant bundle $\det E$,
- A flat connection ∇ on E,
- A flag $0 = F_0 \subset F_1 \subset \cdots \subset F_K = E$ of subbundles of E,

such that

- Each F_n is holomorphic (with respect to the holomorphic structure $\bar\partial_\nabla$),
- If ψ is a section of F_n then $\nabla\psi$ lies in the subbundle $F_{n+1} \otimes K_C \subset E \otimes K_C$,
- $F_\bullet$ is transverse to ∇-flat sections, i.e. the induced linear map

$$\bar\nabla : F_n/F_{n-1} \to F_{n+1}/F_n \otimes K_C \tag{3.33}$$

  is an isomorphism of line bundles, for $1 \le n \le K-1$.

A flat holomorphic bundle (E,∇) can admit at most one flag $F_\bullet$ satisfying the properties above. Thus an $SL(K,\mathbb{C})$-oper is a special sort of flat $SL(K,\mathbb{C})$-connection, and in fact, $SL(K,\mathbb{C})$-opers form a holomorphic Lagrangian subspace in the moduli space of flat $SL(K,\mathbb{C})$-connections. For more background on opers see e.g. [FBZ04, Wen14, Dal08].

3.2.10 A construction of opers

We now describe a concrete construction of $SL(K,\mathbb{C})$-opers which will be particularly convenient for our purposes. We first describe a 1-parameter family of bundles $E_\hbar$ ($\hbar \in \mathbb{C}$), equipped with holomorphic structures $\bar\partial_{E_\hbar}$ and holomorphic flags $F_{\hbar,\bullet}$. Then for any $\mathbf{u} \in \mathcal{B}$ we will construct a corresponding 1-parameter family of connections $\nabla_{\hbar,\mathbf{u}}$ ($\hbar \in \mathbb{C}^\times$), compatible with the holomorphic structures and flags, so that $(E_\hbar, \nabla_{\hbar,\mathbf{u}}, F_{\hbar,\bullet})$ is a 1-parameter family of opers:

Proposition 3.2.7. *We have the following:*

- *For any $\hbar \in \mathbb{C}$, the transition functions*

$$T_{\hbar,z,z'} = \alpha^H_{z,z'} \exp(\hbar \alpha^{-1}_{z,z'} \partial_z \alpha_{z,z'} X_+). \tag{3.34}$$

  define a holomorphic rank K vector bundle $(E_\hbar, \bar\partial_{E_\hbar})$ over C, carrying a flag $F_{\hbar,\bullet}$, and equipped with a distinguished trivialization for each local coordinate patch (U,z) on C.

- *For any $\hbar \in \mathbb{C}^\times$ and $\mathbf{u} \in \mathcal{B}$, there exists a canonical $SL(K,\mathbb{C})$-oper $(E_\hbar, \nabla_{\hbar,\mathbf{u}}, F_{\hbar,\bullet})$, compatible with the holomorphic structure $\bar\partial_{E_\hbar}$. Relative to the distinguished trivializations of $E_\hbar$ on patches (U,z) in the atlas given by Fuchsian uniformization, $\nabla_{\hbar,\mathbf{u}}$ is given by the explicit formula (3.45) below.*

The remainder of this section is devoted to the proof of Proposition 3.2.7.

We define $E_\hbar$ concretely by fixing an atlas of coordinate charts (U,z) and giving transition functions. When $\hbar = 0$, E_0 is just the bundle E described by (3.19), with transition functions $\alpha^H_{z,z'}$ as given in (3.20). The transition functions $T_{\hbar,z,z'}$ for $E_\hbar$ are a deformation of this. However, there is still something to check:

Lemma 3.2.8. *The transition functions* (3.34) *obey the cocycle condition*

$$T_{\hbar,z,z''} = T_{\hbar,z',z''}T_{\hbar,z,z'}. \tag{3.35}$$

Proof. We will exhibit an alternative representation

$$T_{\hbar,z,z'} = M_{\hbar,z'}\alpha^H_{z,z'}M^{-1}_{\hbar,z}, \tag{3.36}$$

from which the cocycle condition (3.35) is immediate.

Represent any fixed metric g on C locally as

$$g = \lambda_z^2 \mathrm{d}z\mathrm{d}\bar{z}, \tag{3.37}$$

and let

$$f_z = \partial_z \log \lambda_z. \tag{3.38}$$

Then $\lambda_{z'} = \lambda_z|\alpha_{z,z'}|^2$, whence

$$f_{z'} = (\partial_z \alpha_{z,z'})\alpha_{z,z'} + \alpha^2_{z,z'}f_z. \tag{3.39}$$

Now define

$$M_{\hbar,z} = \exp(\hbar f_z X_+). \tag{3.40}$$

Then we compute directly

$$\begin{aligned} M_{\hbar,z'}\alpha^H_{z,z'}M^{-1}_{\hbar,z} &= \exp(\hbar f_{z'}X_+)\alpha^H_{z,z'}\exp(-\hbar f_z X_+) \qquad (3.41)\\ &= \alpha^H_{z,z'}\exp(\hbar f_{z'}\alpha^{-2}_{z,z'}X_+)\exp(-\hbar f_z X_+) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_{z,z'}^{H} \exp(\hbar \alpha_{z,z'}^{-1} \partial_z \alpha_{z,z'} X_+) \\
&= T_{\hbar,z,z'}
\end{aligned}
$$

where the second equality uses the relation

$$\alpha^{-H} X_+ \alpha^H = \alpha^{-2} X_+, \tag{3.42}$$

obtained by exponentiating (3.12) and the third uses (3.39). □

We have now shown that the transition functions $T_{\hbar,z,z'}$ determine a vector bundle $E_\hbar$ over C. Moreover, they are holomorphic in the distinguished local trivializations, so E has the holomorphic structure $\bar{\partial}_{E_\hbar} = \bar{\partial}$. (In other words, the distinguished local trivializations are holomorphic.) Note also that $T_{\hbar,z,z'}$ is an upper-triangular matrix, so it preserves the flag $F_{\hbar,\bullet}$, where $F_{\hbar,n}$ is spanned by the first n basis vectors, and this flag is defined globally.

Although $E_0 \not\cong E_\hbar$ when $\hbar \neq 0$, it turns out that all the other $E_\hbar$ are isomorphic:

Proposition 3.2.9. *For any $\lambda \in \mathbb{C}^\times$ and $\hbar \in \mathbb{C}$, there is an isomorphism $E_\hbar \xrightarrow{\sim} E_{\lambda^2\hbar}$ given by λ^H in distinguished local trivializations.*

Proof. We simply note that by(3.42) and (3.34),

$$\lambda^H T_{z,z',\hbar} = T_{z,z',\lambda^2\hbar} \lambda^H. \tag{3.43}$$

□

We now finally describe the connection $\nabla_{\hbar,\mathbf{u}}$ on $E_\hbar$. For this purpose it is convenient to restrict the choice of coordinate systems. We fix a complex projective structure on C, i.e. an atlas of coordinate charts (U, z) with coordinates differing by Möbius transformations,

$$z' = \frac{az+b}{cz+d}, \qquad ad - bc = 1. \tag{3.44}$$

The particular complex projective structure we choose is the one coming from Fuchsian uniformization, i.e. the realization of C as a quotient of the upper half-plane by a subgroup $\Gamma \subset SL(2,\mathbb{R})$.

For each coordinate system in this smaller atlas, we have a distinguished trivialization of $E_\hbar$. Relative to these trivializations, the oper connection equals

$$\nabla_{\hbar,\mathbf{u},z} = \mathrm{d} + \frac{1}{\hbar}\varphi_{\mathbf{u},z}, \tag{3.45}$$

where (as we have stated before),

$$\varphi_{\mathbf{u},z} = \left(X_- + \sum_{n=1}^{K-1} P_{n+1,z} X_n \right) \mathrm{d}z. \tag{3.46}$$

Once again there is something to check:

Lemma 3.2.10. *The formula (3.45) defines a global connection in $E_\hbar$.*

Proof. We must check that

$$T_{\hbar,z,z'} \circ \nabla_{\hbar,\mathbf{u},z} \circ T^{-1}_{\hbar,z,z'} = \nabla_{\hbar,\mathbf{u},z'} \tag{3.47}$$

when z and z' are related by (3.44). We compute the LHS directly, writing $\alpha = \alpha_{z,z'}$ for simplicity. It is a sum of three terms. The first is

$$\begin{aligned} & \alpha^H \exp(\hbar\alpha^{-1}\partial_z\alpha X_+) \circ \mathrm{d} \circ \exp(-\hbar\alpha^{-1}\partial_z\alpha X_+)\alpha^{-H} \\ = \; & \mathrm{d} + \left[\alpha^H \left((-\hbar\partial_z^2 \log\alpha) X_+\right) \alpha^{-H} + (-\partial_z \log\alpha) H\right] \mathrm{d}z \\ = \; & \mathrm{d} + \left[(-\hbar\alpha^2\partial_z^2 \log\alpha) X_+ + (-\partial_z \log\alpha) H\right] \mathrm{d}z. \end{aligned} \tag{3.48}$$

Next is

$$\begin{aligned} & \hbar^{-1}\alpha^H \exp(\hbar\alpha^{-1}\partial_z\alpha X_+) X_- \exp(-\hbar\alpha^{-1}\partial_z\alpha X_+)\alpha^{-H} \\ = \; & \hbar^{-1}\alpha^H (X_- + (\hbar\partial_z \log\alpha)[X_+, X_-] + \frac{1}{2}(\hbar\partial_z \log\alpha)^2[X_+, [X_+, X_-]])\alpha^{-H} \\ = \; & \hbar^{-1}\alpha^H (X_- + (\hbar\partial_z \log\alpha) H - (\hbar\partial_z \log\alpha)^2 X_+)\alpha^{-H} \\ = \; & \hbar^{-1}\alpha^{-2} X_- + (\partial_z \log\alpha) H - \hbar\alpha^2(\partial_z \log\alpha)^2 X_+. \end{aligned} \tag{3.49}$$

The transformation for the other terms in $\hbar^{-1}\varphi_{\mathbf{u},z}$ is simpler since they commute with X_+, and we obtain

$$\begin{aligned} & \hbar^{-1}\alpha^H \exp(\hbar\alpha^{-1}\partial_z\alpha X_+) \left(\sum_{n=1}^{K-1} P_{n+1,z} X_n\right) \exp(-\hbar\alpha^{-1}\partial_z\alpha X_+)\alpha^{-H} \\ = \; & \hbar^{-1}\alpha^H \left(\sum_{n=1}^{K-1} P_{n+1,z} X_n\right) \alpha^{-H} \\ = \; & \hbar^{-1} \sum_{n=1}^{K-1} P_{n+1,z} \alpha^{2n} X_n. \end{aligned} \tag{3.50}$$

Combining all these terms, the terms proportional to H cancel nicely and we get the desired result $\nabla_{\hbar,\mathbf{u},z'}$, *except* for an extra term εX_+, where

$$\varepsilon = -\hbar((\partial_z\alpha)^2 + \alpha^2\partial_z^2 \log\alpha). \tag{3.51}$$

It is precisely at this point where we have to use the restriction of the coordinate atlas. Indeed, for the transformations (3.44),

$$\alpha = \pm(cz + d), \tag{3.52}$$

so (3.51) vanishes in this case. $\square$

Finally note that the explicit formulas (3.45) and (3.11) say $\nabla_{\hbar,\mathbf{u},z}$ has nowhere-vanishing entries on the first sub-diagonal. This is equivalent to saying that $\nabla_{\hbar,\mathbf{u}}$ obeys the transversality condition in Definition 3.2.8, and completes the proof that $(E_\hbar, \nabla_{\hbar,\mathbf{u}}, F_{\hbar,\bullet})$ is an $SL(K,\mathbb{C})$-oper, thus completing the proof of Proposition 3.2.7.

3.2.11 $SL(K,\mathbb{C})$-opers and differential operators

This section is not used directly in the rest of the paper. Its purpose is to recall the sense in which $SL(K,\mathbb{C})$-opers are equivalent to certain K-th order linear scalar differential operators. This construction is well known; we include it here just to spell out its relation with the description of opers in Proposition 3.2.7, as connections on $E_\hbar$.

When ψ is a holomorphic section of $\mathcal{L}^{1-K}$, let $\psi^{[K-1]}$ denote the $(K-1)$-jet of ψ, which is a holomorphic section of the jet bundle $J^{K-1}(\mathcal{L}^{1-K})$.

Proposition 3.2.11. *Fix* $\mathbf{u} \in \mathcal{B}$. *There exists a unique holomorphic isomorphism*

$$\Phi_{\mathbf{u}} : J^{K-1}(\mathcal{L}^{1-K}) \xrightarrow{\sim} E_\hbar \tag{3.53}$$

such that

- *$\nabla_{\hbar,\mathbf{u}}(\Phi_{\mathbf{u}}(\psi^{[K-1]}))$ is valued in the holomorphic line subbundle $\mathcal{L}^{K-1} \otimes K_C \simeq \mathcal{L}^{K+1}$ of $E_\hbar \otimes K_C$,*

- *$\Phi_{\mathbf{u}}$ descends to a map between the 1-dimensional quotient $\mathcal{L}^{1-K}$ of $J^{K-1}(\mathcal{L})$ and the 1-dimensional quotient $E_\hbar / F_{K-1} \simeq \mathcal{L}^{1-K}$; this map is $\hbar$ times the identity.*

The map

$$\mathcal{D}_{\hbar,\mathbf{u}} : \psi \mapsto \nabla_{\hbar,\mathbf{u}}(\Phi_{\mathbf{u}}(\psi^{[K-1]})) \tag{3.54}$$

is a linear differential operator of order K, mapping $\mathcal{L}^{1-K} \to \mathcal{L}^{K+1}$.

This becomes much more concrete when we write $\Phi_{\mathbf{u}}$ relative to the distinguished local trivializations in the Fuchsian atlas. For instance, when $K = 2$, we have

$$\left[\partial_z + \frac{1}{\hbar}\begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} -\hbar^2 \psi'_z \\ \hbar \psi_z \end{pmatrix} = \begin{pmatrix} -\hbar^2 \psi''_z + P_2 \psi_z \\ 0 \end{pmatrix}. \tag{3.55}$$

This equation implies that

$$\Phi_{\mathbf{u}}(\psi^{[1]}) = \begin{pmatrix} -\hbar^2 \psi'_z \\ \hbar \psi_z \end{pmatrix}. \tag{3.56}$$

The 0 in the bottom component of the RHS of (3.55) says $\nabla \Phi_{\mathbf{u}}(\psi^{[1]})$ is valued in the subbundle $\mathcal{L} \otimes K_C$; this condition determines $\Phi_{\mathbf{u}}$ up to a constant multiple which is fixed by requiring that the bottom component of $\Phi_{\mathbf{u}}(\psi^{[1]})$ is exactly $\hbar \psi_z$.

Thus we can read off from the top component of the RHS of (3.55) that $\mathcal{D}_{\hbar,\mathbf{u}}$ is represented locally by

$$\mathcal{D}_{\hbar,\mathbf{u}} = -\hbar^2 \partial_z^2 + P_2. \tag{3.57}$$

Similarly, for $K = 3$, the analogue of (3.55) is

$$\left[\partial_z + \frac{1}{\hbar}\begin{pmatrix} 0 & \sqrt{2}P_2 & P_3 \\ \sqrt{2} & 0 & \sqrt{2}P_2 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right]\begin{pmatrix} \frac{\hbar^3}{2}\psi_z'' - \hbar P_2\psi_z \\ -\frac{\hbar^2}{\sqrt{2}}\psi_z' \\ \hbar\psi_z \end{pmatrix} = \begin{pmatrix} \frac{\hbar^3}{2}\psi_z''' - 2\hbar P_2\psi_z' - \hbar P_2'\psi_z + P_3\psi_z \\ 0 \\ 0 \end{pmatrix}, \tag{3.58}$$

which says that we have

$$\Phi_{\mathbf{u}}(\psi^{[2]}) = \begin{pmatrix} \frac{\hbar^3}{2}\psi_z'' - \hbar P_2\psi_z \\ -\frac{\hbar^2}{\sqrt{2}}\psi_z' \\ \hbar\psi_z \end{pmatrix} \tag{3.59}$$

and

$$\mathcal{D}_{\hbar,\mathbf{u}} = \frac{\hbar^3}{2}\partial_z^3 - 2\hbar P_2\partial_z - \hbar P_2' + P_3. \tag{3.60}$$

Note that $\Phi_{\mathbf{u}}$ depends on $\mathbf{u}$ through the term P_2 in (3.59). The $\mathbf{u}$ dependence of $\mathcal{D}_{\hbar,\mathbf{u}}$ is thus more complicated than one might naively guess: we already see that P_2' appears in (3.60) despite the fact that only the P_n and not their derivatives appear in the formula (3.45) defining $\nabla_{\hbar,\mathbf{u}}$.

3.3 Gaiotto's conjecture and proof, for $G = SL(K, \mathbb{C})$

3.3.1 Opers and the Hitchin component: Gaiotto's conjecture

Fix an $SL(K,\mathbb{C})$-Higgs bundle $(E, \bar{\partial}, \varphi)$ on C. We then have the 2-parameter family of flat connections in E (3.9) depending on $\zeta \in \mathbb{C}^\times$ and $R \in \mathbb{R}^+$, where $h(R)$ is the harmonic metric guaranteed by Theorem 3.2.2 and $D = D_{h(R)}$.

We are going to consider the limits of certain 1-parameter subfamilies, obtained by taking $R \to 0$ and $\zeta \to 0$ simultaneously while holding their ratio fixed. In other words, fix some $\hbar \in \mathbb{C}^\times$ and set $\zeta = R\hbar$: then (3.9) becomes

$$\nabla_{R,\hbar} = \hbar^{-1}\varphi + D_{h(R)} + \hbar R^2 \varphi^{\dagger_{h(R)}}. \tag{3.61}$$

In [Gai14], Gaiotto proposed (and gave considerable evidence for):

Conjecture 3.3.1. *Suppose the $SL(K,\mathbb{C})$-Higgs bundle $(E,\bar{\partial},\varphi)$ is in the Hitchin component, and fix some $\hbar \in \mathbb{C}^\times$. Then as $R \to 0$ the connections $\nabla_{R,\hbar}$ converge to a connection $\nabla_{0,\hbar}$ in E, and there exists a flag $F_\bullet$ in E such that $(E,\nabla_{0,\hbar},F_\bullet)$ is an $SL(K,\mathbb{C})$-oper.*

We will prove the following explicit version of Conjecture 3.3.1:

Theorem 3.3.2. *Fix any $\mathbf{u} \in \mathcal{B}$. Let $(E,\bar{\partial},\varphi_{\mathbf{u}})$ be the corresponding Higgs bundle in the Hitchin component, and let $h(R,\mathbf{u})$ be the family of harmonic metrics on E solving the rescaled Hitchin equation (3.8). Let $F_\bullet$ be the flag*

$$F_n = \bigoplus_{i=1}^{n} \mathcal{L}^{K+1-2i} \subset E. \tag{3.62}$$

Fix $\hbar \in \mathbb{C}^\times$ and let

$$\nabla_{R,\hbar,\mathbf{u}} = \hbar^{-1}\varphi_{\mathbf{u}} + D_{h(R,\mathbf{u})} + \hbar R^2 \varphi_{\mathbf{u}}^{\dagger_{h(R,\mathbf{u})}}. \tag{3.63}$$

Then, as $R \to 0$ the flat connections $\nabla_{R,\hbar,\mathbf{u}}$ converge to a flat connection

$$\nabla_{0,\hbar,\mathbf{u}} = \hbar^{-1}\varphi_{\mathbf{u}} + D_{h_\natural} + \hbar \varphi_{\mathbf{0}}^{\dagger_{h_\natural}}, \tag{3.64}$$

and $(E,\nabla_{0,\hbar,\mathbf{u}},F_\bullet)$ is an $SL(K,\mathbb{C})$-oper, equivalent to the $SL(K,\mathbb{C})$-oper $(E_\hbar,\nabla_{\hbar,\mathbf{u}},F_{\hbar,\bullet})$ of Proposition 3.2.7.

We emphasize that the harmonic metric $h(R)$ depends on R, and indeed (as we will see) $h(R)$ diverges as $R \to 0$. In particular, we cannot simply drop the last term of (3.63) in the $R \to 0$ limit, despite the explicit prefactor R^2; it survives to become the last term of (3.64), and is ultimately responsible for the deformation of the holomorphic structure as a function of $\hbar$.

3.3.2 Proof of Gaiotto's conjecture for $G = SL(2, \mathbb{C})$

The case $K = 2$ of Theorem 3.3.2 is notationally simpler, and contains the main ideas, so we do it separately.

Fix a coordinate patch (U, z) on C, and the corresponding distinguished trivialization of E. Our first aim is to write an explicit local formula, (3.72) below, for the family of connections (3.63) in E.

First recall that the decomposition

$$E = \mathcal{L} \oplus \mathcal{L}^{-1} \tag{3.65}$$

is orthogonal for $h(R, \mathbf{u})$ [CL14]. Furthermore, using that $\mathcal{L}^2 \simeq K_C$, $h(R, \mathbf{u})$ is induced from a Hermitian metric $g(R, \mathbf{u})$ on C. In the local patch (U, z),

$$g(R, \mathbf{u}) = \lambda(R, \mathbf{u}; z)^2 \mathrm{d}z \mathrm{d}\bar{z} \tag{3.66}$$

for some real-valued function λ, and

$$h(R, \mathbf{u}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{3.67}$$

We now write the Chern connection D_h explicitly. Since the distinguished trivializations are holomorphic, the $(0,1)$ part $\bar{\partial}_{D_h}$ is simply represented by $\bar{\partial}$. The $(1,0)$ part ∂_{D_h} is then determined by unitarity with respect to h, which gives

$$\partial_{D_h} = \partial - \begin{pmatrix} \partial \log \lambda & 0 \\ 0 & -\partial \log \lambda \end{pmatrix}, \tag{3.68}$$

so altogether the Chern connection is

$$D_h = \mathrm{d} - \begin{pmatrix} \partial \log \lambda & 0 \\ 0 & -\partial \log \lambda \end{pmatrix}. \tag{3.69}$$

Next, note that fixing $\mathbf{u} \in \mathcal{B}$ just means fixing a holomorphic quadratic differential $\phi_2 \in H^0(C, K_C^2)$. Locally,

$$\phi_2 = P \, \mathrm{d}z^2 \tag{3.70}$$

where P is a holomorphic function on U, and

$$\varphi_{\mathbf{u}} = \begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix} \mathrm{d}z, \qquad \varphi_{\mathbf{u}}^{\dagger_h} = \begin{pmatrix} 0 & \lambda^2 \\ \lambda^{-2}\overline{P} & 0 \end{pmatrix} \mathrm{d}\bar{z}. \tag{3.71}$$

Combining all this gives the desired explicit representation,

$$\nabla_{R,\hbar,\mathbf{u}} = \mathrm{d} + \frac{1}{\hbar} \begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix} \mathrm{d}z - \begin{pmatrix} \partial \log \lambda & 0 \\ 0 & -\partial \log \lambda \end{pmatrix} + R^2 \hbar \begin{pmatrix} 0 & \lambda^2 \\ \lambda^{-2}\overline{P} & 0 \end{pmatrix} \mathrm{d}\bar{z}. \tag{3.72}$$

The flatness of $\nabla_{R,\hbar,\mathbf{u}}$ is equivalent to the fact that h is the harmonic metric. Thus computing the curvature of $\nabla_{R,\hbar,\mathbf{u}}$ directly from (3.72) gives the harmonicity condition, as an equation for λ (also written e.g. in [Hit87, GMN09]):

$$\partial_{\bar{z}} \partial_z \log \lambda - R^2 (\lambda^2 - \lambda^{-2} |P|^2) = 0. \tag{3.73}$$

We must understand the behavior of the solution $\lambda = \lambda(R, \mathbf{u})$ as $R \to 0$.

To get some intuition, first consider the special case $P = 0$ (corresponding to the quadratic differential $\phi_2 = 0$.) In this case (3.73) specializes to

$$\partial_{\bar{z}} \partial_z \log \lambda - R^2 \lambda^2 = 0, \tag{3.74}$$

which simply says that the metric $g(R, \mathbf{0})$ of (3.66) has constant curvature $-4R^2$. Hence $g(R, \mathbf{0}) = \frac{1}{R^2} g_\natural$ where $g_\natural$ is the unique metric with constant curvature -4, and

$$\lambda(R, \mathbf{0}) = \frac{\lambda_\natural}{R}. \tag{3.75}$$

In the general case where P is not necessarily 0, we use $g_\natural$ as a background metric and write

$$g(R, \mathbf{u}) = \frac{\mathrm{e}^{2f}}{R^2} g_\natural \tag{3.76}$$

where f is a real-valued function on C. We claim that as $R \to 0$,

$$\partial_z^a \partial_{\bar{z}}^b f = O(R^4) \quad \text{for all} \quad a, b \geq 0, \tag{3.77}$$

(so in particular, $f = O(R^4)$).

To prove this, first rewrite (3.73) in terms of the Laplacian for $g_\natural$, $\Delta_{g_\natural} = \frac{4}{\lambda^2}\partial_{\bar{z}}\partial_z = \frac{1}{\lambda^2}\left(\partial_x^2 + \partial_y^2\right)$:

$$N(f, R) = \Delta_{g_\natural} f + 4\left(1 - \mathrm{e}^{2f} + R^4 |P|^2 \mathrm{e}^{-2f}\right) = 0. \tag{3.78}$$

The maximum principle shows that for any $R \geq 0$, there exists at most one function f such that $N(f, R) = 0$. In fact, the method of upper and lower solutions shows that there is exactly one solution, but of course we already know this when $R > 0$ from the existence and uniqueness of harmonic metrics, and $f = 0$ is a solution when $R = 0$.

The linearization at $R = 0$ is

$$DN|_{(0,0)}(\dot{f}, 0) = \left(\Delta_{g_\natural} - 8\right)\dot{f}, \tag{3.79}$$

and this is an isomorphism as a map $\mathcal{C}^{k+2,\alpha}(C) \to \mathcal{C}^{k,\alpha}(C)$ for any $k \geq 0$ and $\alpha \in (0,1)$. Note also that N is a $\mathcal{C}^\infty$ mapping from a neighborhood of 0 in $\mathcal{C}^{k+2,\alpha}(C) \times \mathbb{R}$ to $\mathcal{C}^{k,\alpha}(C)$. The Banach space implicit function theorem now gives the existence of a $\mathcal{C}^\infty$ map $\Psi : [0, R_0) \to \mathcal{C}^{k+2,\alpha}(C)$ such that $N(\Psi(R), R) = 0$ for $0 \leq R < R_0$, and

$\Psi(0) = 0$. From the uniqueness it follows that Ψ is independent of k and α, so that $\Psi(R)$ is a $\mathcal{C}^\infty$ function on C for each $R \geq 0$, and in fact, $(z, R) \mapsto \Psi(R)(z)$ lies in $\mathcal{C}^\infty(C \times [0, R_0))$. We can say even more: since all data in N is real analytic, the real analytic version of the implicit function theorem [KP13] shows that Ψ is real analytic in R. Finally, by the uniqueness of harmonic metrics, $\Psi(R)$ must agree with the desired f when $R > 0$.

The upshot of the last paragraph is that we may expand f in a Taylor series around $R = 0$,

$$f = Rf_1 + R^2 f_2 + \cdots \tag{3.80}$$

Substituting this series into (3.78), we see that $f_1 = f_2 = f_3 = 0$, and hence we get (3.77) as desired.

It follows that as $R \to 0$ we have

$$\lambda = \frac{\lambda_\natural}{R} + O(R^3). \tag{3.81}$$

Substituting this in (3.72), we see that as $R \to 0$, $\nabla_{R,\hbar,\mathbf{u}}$ converges to

$$\nabla_{0,\hbar,\mathbf{u}} = \mathrm{d} + \frac{1}{\hbar}\begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix} \mathrm{d}z - \begin{pmatrix} \partial \log \lambda_\natural & 0 \\ 0 & -\partial \log \lambda_\natural \end{pmatrix} + \hbar \begin{pmatrix} 0 & \lambda_\natural^2 \\ 0 & 0 \end{pmatrix} \mathrm{d}\bar{z}. \tag{3.82}$$

This is the desired (3.64).

It is instructive to see directly that $(E, \nabla_{0,\hbar,\mathbf{u}}, F_\bullet)$ is an $SL(2,\mathbb{C})$-oper, where $F_\bullet$ is the flag

$$0 \subset \mathcal{L} \subset E. \tag{3.83}$$

For this the key is the lower left entry $\frac{1}{\hbar}\mathrm{d}z$ in (3.82) maps $\mathcal{L} \to \mathcal{L}^{-1} \otimes K_C$. The two salient facts about this are:

- Its $(0,1)$ part is trivial, so $\mathcal{L}$ is a holomorphic subbundle of $(E, \bar{\partial}_{\nabla_{0,\hbar,\mathbf{u}}})$;
- Its $(1,0)$ part is nowhere vanishing, i.e., $\bar{\nabla}_{0,\hbar,\mathbf{u}} : \mathcal{L} \to (E/\mathcal{L}) \otimes K_C$ is an isomorphism of line bundles.

These conditions say precisely that $(E, \nabla_{0,\hbar,\mathbf{u}}, F_\bullet)$ is an $SL(2,\mathbb{C})$-oper.

Finally we want to show that $(E, \nabla_{0,\hbar,\mathbf{u}}, F_\bullet)$ is equivalent to $(E_\hbar, \nabla_{\hbar,\mathbf{u}}, F_{\hbar,\bullet})$, the $SL(2,\mathbb{C})$-oper of Proposition 3.2.7. Comparing (3.82) to the desired form (3.45), we see that we need to change our local trivializations by a gauge transformation which eliminates the last two terms in (3.82), i.e. by a matrix of the form

$$M_{\hbar,z} = \begin{pmatrix} 1 & \hbar\beta \\ 0 & 1 \end{pmatrix}, \tag{3.84}$$

where $\partial_{\bar{z}}\beta = \lambda_\natural^2$. Because of the equation (3.30) for $\lambda_\natural$, there is a natural candidate, $\beta = \partial_z \log \lambda_\natural$, leading to

$$M_{\hbar,z} = \begin{pmatrix} 1 & \hbar\partial_z \log \lambda_\natural \\ 0 & 1 \end{pmatrix}. \tag{3.85}$$

Relative to the new local trivializations, the transition maps from patches (U,z) to (U',z') become the ones we wrote in (3.36); these are indeed the transition maps of $E_\hbar$. What remains is to compute $\nabla_{0,\hbar,\mathbf{u}}$ in the new trivializations. Computing directly $M_{\hbar,z} \circ \nabla_{0,\hbar,\mathbf{u}} \circ M_{\hbar,z}^{-1}$ from (3.82), (3.85) we obtain

$$\mathrm{d} + \frac{1}{\hbar}\begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix}\mathrm{d}z + \hbar\begin{pmatrix} 0 & \frac{2(\partial_z\lambda_\natural)^2 - \lambda_\natural\partial_z^2\lambda_\natural}{\lambda_\natural^2} \\ 0 & 0 \end{pmatrix}\mathrm{d}z. \tag{3.86}$$

If our coordinate patch (U,z) is in the atlas given by Fuchsian uniformization, then the explicit form of the hyperbolic metric in the upper half-plane gives $\lambda_\natural = \frac{\mathrm{i}}{z-\bar{z}}$, and then the last term in (3.86) vanishes. Thus it reduces to the desired (3.45). This finishes the proof of Theorem 3.3.2 in case $K = 2$.

3.3.3 Proof of Gaiotto's conjecture for $G = SL(K, \mathbb{C})$

Now we prove Theorem 3.3.2 in full generality. The proof is essentially the same as for $K = 2$, with three differences:

- The notation becomes less transparent, because we cannot write everything in terms of explicit 2×2 matrices anymore.
- The harmonic metrics $h(R, \mathbf{u})$ are no longer determined by a single function on C, so we have to study a coupled system of equations instead of a single scalar equation.
- The harmonic metrics $h(R, \mathbf{u})$ may not be diagonal in the distinguished trivializations.

As in the case $K = 2$, the main technical issue is to control the the harmonic metric $h(R, \mathbf{u})$ in the limit $R \to 0$. We will show that in this limit $h(R, \mathbf{u})$ approaches the natural metric $h_\natural(R)$. More precisely:

Lemma 3.3.3. *If we write*

$$h(R, \mathbf{u}) = h_\natural(R) \mathrm{e}^{\chi(R, \mathbf{u})}, \tag{3.87}$$

then $\chi(R, \mathbf{u}) \in \mathrm{End}(E)$ *satisfies*

$$\chi(R, \mathbf{u}) = O(R^4). \tag{3.88}$$

Proof. First some notation: for $\mathbf{u} = (\phi_2, \dots, \phi_K) \in \mathcal{B}$ and $\alpha \in \mathbb{R}_+$, we let

$$\alpha \mathbf{u} = (\alpha^2 \phi_2, \dots, \alpha^K \phi_K) \in \mathcal{B}. \tag{3.89}$$

Now define

$$N_{\mathbf{u}}(\chi, R) = \left[\bar{\partial}_E, \mathrm{e}^{-\chi}\partial_E^{h_\natural}\mathrm{e}^{\chi}\right] + \left[\varphi_{\mathbf{u}}, \mathrm{e}^{-\chi}\varphi_{R^2\mathbf{u}}^{\dagger_{h_\natural}}\mathrm{e}^{\chi}\right]. \tag{3.90}$$

For any fixed R, this is an operator

$$N_{\mathbf{u}}(\cdot, R) : \Omega^0(\mathrm{End}_Q E) \to \Omega^2(\mathrm{End}_Q E). \tag{3.91}$$

We proceed in steps:

1. For any $R > 0$, $N_{\mathbf{u}}(\chi, R) = 0$ if, and only if, $h_\natural(R)\mathrm{e}^{\chi}$ is the harmonic metric for $\varphi_{\mathbf{u}}$ with parameter R.

2. $N_{\mathbf{u}}(0, 0) = 0$.

3. The linearization $D_\chi N_{\mathbf{u}}|_{(0,0)}$ is bijective.

4. There exists a smooth function $\chi(R, \mathbf{u})$ for $R \in [0, R_0)$ such that

$$N_{\mathbf{u}}(\chi(R, \mathbf{u}), R) = 0. \tag{3.92}$$

5. The first nonzero term in this Taylor series appears at order R^4.

For (1) we compute directly. The curvature of the Chern connection $D(h)$ for the metric $h = h_\natural(R)\mathrm{e}^{\chi} = h_\natural \tau_R \mathrm{e}^{\chi}$ is

$$F_{D(h)} = \left[\bar{\partial}_E, \mathrm{e}^{-\chi}\tau_R^{-1}\partial_E^{h_\natural}\tau_R\mathrm{e}^{\chi}\right] = \left[\bar{\partial}_E, \mathrm{e}^{-\chi}\partial_E^{h_\natural}\mathrm{e}^{\chi}\right], \tag{3.93}$$

while

$$\varphi^{\dagger_h} = \mathrm{e}^{-\chi}\tau_R^{-1}\varphi^{\dagger_{h_\natural}}\tau_R\mathrm{e}^{\chi} = R^{-2}\mathrm{e}^{-\chi}\varphi_{R^2\mathbf{u}}^{\dagger_{h_\natural}}\mathrm{e}^{\chi}. \tag{3.94}$$

Combining these gives

$$F_{D(h)} + R^2[\varphi, \varphi^{\dagger_h}] = N_{\mathbf{u}}(\chi, R). \tag{3.95}$$

For (2), observe first that

$$N_{\mathbf{u}}(0,0) = \left[\bar{\partial}_E, \partial_E^{h_\natural}\right] + \left[\varphi_{\mathbf{u}}, \varphi_{\mathbf{0}}^{\dagger_{h_\natural}}\right]. \tag{3.96}$$

This would vanish if $\varphi_{\mathbf{u}}$ were replaced by $\varphi_{\mathbf{0}}$ since $h_\natural$ is the harmonic metric for the Higgs field $\varphi_{\mathbf{0}}$. However, the difference $\varphi_{\mathbf{u}} - \varphi_{\mathbf{0}}$ is a sum of terms X_n, all of which commute with $\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}$, since $\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}$ is proportional to X_+, and $[X_n, X_+] = 0$ by (3.14).

For (3) we compute that

$$\mathcal{L}_{\mathbf{u}}\dot{\chi} := D_\chi N_{\mathbf{u}}|_{(0,0)}\,\dot{\chi} = \bar{\partial}_E \partial_E^{h_\natural}\dot{\chi} + \left[\varphi_{\mathbf{u}}, \left[\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}, \dot{\chi}\right]\right]. \tag{3.97}$$

We wish to show this operator has trivial kernel. First consider the case $\mathbf{u} = \mathbf{0}$. Using the L^2 pairing induced by $h_\natural$, we have

$$\langle \dot{\chi}, \mathcal{L}_{\mathbf{0}}\dot{\chi}\rangle = \|\partial_E^{h_\natural}\dot{\chi}\|^2 + \left\|\left[\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}, \dot{\chi}\right]\right\|^2. \tag{3.98}$$

By Lemma 3.2.5, the second term on the right is strictly positive if $\dot{\chi} \neq 0$, so $\mathcal{L}_{\mathbf{0}}$ has trivial kernel. It can be deformed amongst elliptic operators to the self-adjoint operator $\bar{\partial}_E \partial_E^{h_\natural}$, and hence has index zero, which means that $\mathcal{L}_{\mathbf{0}}$ is also surjective, and hence an isomorphism $\mathcal{C}^{k+2,\alpha} \to \mathcal{C}^{k,\alpha}$ for any $k \geq 0$.

We now extend this to a statement about $DN_{\mathbf{u}}(\cdot, 0)$. To this end, we use the grading on $\mathrm{End}_Q E$ where we say that a matrix has grade k if, in the distinguished local trivializations, its nonzero entries are k steps above the diagonal. Notice that $\mathcal{L}_{\mathbf{0}}$ preserves this grading. We have

$$\mathcal{L}_{\mathbf{u}}(\cdot) - \mathcal{L}_{\mathbf{0}}(\cdot) = [\varphi_{\mathbf{u}} - \varphi_{\mathbf{0}}, [\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}, \cdot]] \tag{3.99}$$

and this strictly increases the grading since both $\varphi_{\mathbf{u}} - \varphi_{\mathbf{0}}$ and $\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}$ are strictly upper triangular. It follows from this that $\mathcal{L}_{\mathbf{u}}$ also has trivial kernel. (Indeed, given operators A, B where A preserves a grading and B increases it, $(A+B)v = 0$ implies that A annihilates the lowest-grade component of v.) By the same remarks as above, it is also surjective.

To obtain (4) we can apply the implicit function theorem exactly as in the case $K = 2$ above to deduce the existence of a smooth function $\chi(R, \mathbf{u})$ such that $N_{\mathbf{u}}(\chi(R, \mathbf{u}), R) \equiv 0$ for $0 \leq R \leq R_0$. As before, this solution is real analytic in R and z jointly.

Finally, for (5) we simply plug the Taylor series

$$\chi(R, \mathbf{u}) = \sum_{n=1}^{\infty} R^n \chi_n(\mathbf{u}) \tag{3.100}$$

into (3.92), and expand in powers of R. From (3.90) we have

$$N_{\mathbf{u}}(\chi, R) = N_{\mathbf{u}}(\chi, 0) + O(R^4) \tag{3.101}$$

Thus at order R^1 we have to solve

$$\mathcal{L}_{\mathbf{u}}(\chi_1) = 0, \tag{3.102}$$

which we have already seen implies $\chi_1 = 0$. Similarly at orders R^2 and R^3 we get $\chi_2 = \chi_3 = 0$. This finishes the proof. $\square$

Now we are ready to prove Theorem 3.3.2. We just substitute $h(R, \mathbf{u}) = h_\natural(R)\mathrm{e}^{\chi(R,\mathbf{u})}$ in (3.63), obtaining (using (3.94)):

$$\nabla_{R,\hbar,\mathbf{u}} = \hbar^{-1}\varphi_{\mathbf{u}} + \mathrm{e}^{-\chi(R,\mathbf{u})} D_{h_\natural} \mathrm{e}^{\chi(R,\mathbf{u})} + \hbar \mathrm{e}^{-\chi(R,\mathbf{u})} \varphi_{R^2\mathbf{u}}^{\dagger_{h_\natural}} \mathrm{e}^{\chi(R,\mathbf{u})}. \tag{3.103}$$

In the limit $R \to 0$ we have $\chi(R, \mathbf{u}) \to 0$, so this reduces to

$$\nabla_{0,\hbar,\mathbf{u}} = \hbar^{-1}\varphi_{\mathbf{u}} + D_{h_\natural} + \hbar\varphi_{\mathbf{0}}^{\dagger_{h_\natural}}, \tag{3.104}$$

as desired.

To verify that $(E, \nabla_{0,\hbar,\mathbf{u}}, F_\bullet)$ is equivalent to the oper of Proposition 3.2.7, we proceed just as we did in the case $K = 2$: we introduce the matrix

$$M_{\hbar,z} = \exp(\hbar(\partial_z \log \lambda_\natural)X_+) \tag{3.105}$$

and compute directly $M_{\hbar,z} \circ \nabla_{0,\hbar,\mathbf{u}} \circ M_{\hbar,z}^{-1}$, giving

$$\mathrm{d} + \hbar^{-1}\varphi_{\mathbf{u}} + \hbar\varepsilon X_+ \tag{3.106}$$

where the "error term" $\varepsilon = \frac{2(\partial_z\lambda_\natural)^2 - \lambda_\natural\partial_z^2\lambda_\natural}{\lambda_\natural^2}$ vanishes when the local coordinate (U, z) is in the atlas given by Fuchsian uniformization. This completes the proof.

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